

Differential formulas.

Differential and Integral Calculus

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$$(1) \frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

$$(2) \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx} [f(x)]^n = n [f(x)]^{n-1} \cdot \frac{d}{dx} f(x)$$

$$(3) \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{-1}{x^2}$$

$$(4) \frac{d}{dx} (x) = 1$$

$$(5) \frac{d}{dx} (a^x) = a^x \cdot \log a$$

$$(6) \frac{d}{dx} (e^x) = e^x \log e = e^x (1) = e^x$$

$$(7) \frac{d}{dx} (\log x) = \frac{1}{x}$$

$$(8) \frac{d}{dx} (\text{constant}) = 0$$

$$(9) \frac{d}{dx} [f(x) + g(x) - h(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) - \frac{d}{dx} h(x)$$

$$(10) \frac{d}{dx} [k \cdot f(x)] = k \cdot \frac{d}{dx} (f(x))$$

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$$(11) \frac{d}{dx} (\sin x) = \cos x$$

$$(12) \frac{d}{dx} (\cos x) = -\sin x$$

$$(13) \frac{d}{dx} (\tan x) = \sec^2 x$$

$$(14) \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$(15) \frac{d}{dx} (\sec x) = \sec x \tan x$$

$$(16) \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$$

$$(17) \frac{d}{dx} (\sinh x) = \cosh x$$

$$(18) \frac{d}{dx} (\cosh x) = \sinh x$$

$$(19) \frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$$

$$(20) \frac{d}{dx} (\coth x) = -\operatorname{cosech}^2 x$$

$$(21) \frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$(22) \frac{d}{dx} (\operatorname{cosech} x) = -\operatorname{cosech} x \coth x.$$

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$$(23) \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$(24) \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$(25) \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$(26) \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$(27) \frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2-1}}$$

$$(28) \frac{d}{dx} (\operatorname{cosec}^{-1} x) = \frac{-1}{|x| \sqrt{x^2-1}}$$

$$(29) \frac{d}{dx} (u \cdot v) = u \cdot \frac{d}{dx} (v) + v \cdot \frac{d}{dx} (u) \\ = uv' + vu'$$

$$(30) \frac{d}{dx} (uvw) = uv \frac{d}{dx} (w) + uw \frac{d}{dx} (v) + vw \frac{d}{dx} (u) \\ = uvw' + uwv' + vwu'$$

$$(31) \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{d}{dx} (u) - u \cdot \frac{d}{dx} (v)}{v^2} \\ = \frac{vu' - uv'}{v^2}$$

$$\textcircled{1} \quad \frac{d}{dx}(\sec^2 x) = \frac{d}{dx}(\sec^2 x)$$

$$= \frac{d}{dx}(\sec x)^2$$

$$= 2 \cdot (\sec x)^{2-1} \cdot \frac{d}{dx}(\sec x)$$

$$= 2(\sec x) \cdot \sec x \tan x$$

$$= 2\sec^2 x \cdot \tan x.$$

$$* \quad \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \cdot \frac{d}{dx}(f(x))$$

$$* \quad \frac{d}{dx}(\sqrt{f(x)}) = \frac{1}{2\sqrt{f(x)}} \cdot \frac{d}{dx}(f(x))$$

$$* \quad \frac{d}{dx}(\log f(x)) = \frac{1}{f(x)} \cdot \frac{d}{dx}[f(x)]$$

$$* \quad \frac{d}{dx}(\sin f(x)) = \cos f(x) \cdot \frac{d}{dx}(f(x))$$

$$* \quad \frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-1}{[f(x)]^2} \cdot \frac{d}{dx}[f(x)]$$

Trigonometric Formulas :-

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\sin \theta = 2 \sin \theta/2 \cos \theta/2$$

$$\cos 2\theta = 2 \cos^2 \theta - 1 \Rightarrow 1 + \cos 2\theta = 2 \cos^2 \theta$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta \Rightarrow 1 - \cos 2\theta = 2 \sin^2 \theta$$

Note: Partial Differentiation (Unit -1)

$\frac{\partial f}{\partial x} \Rightarrow$ partial derivative of f wrt. x , except x terms in f are all treated as a constant

Ex: ① $f = xy$

p.d wrt to x
Partial Derivative

$$\frac{\partial f}{\partial x} = y \cdot \frac{\partial}{\partial x} (x)$$

$$= y(1)$$

$$= y$$

Similarly, $f = xy$

p.d wrt to y

$$\frac{\partial f}{\partial y} = x \cdot \frac{\partial}{\partial y} (y)$$

$$= x(1)$$

$$= x$$

Similarly, $f = xyz$

p.d wrt to z

$$\frac{\partial f}{\partial z} = 0.$$

②. $f(x, y) = x^3 + 6x^2y - y^4$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Given that,

$$f(x, y) = x^3 + 6x^2y - y^4$$

P.D w.r.to x

$$\frac{\partial f}{\partial x} = 3x^2 + 6y \cdot \frac{\partial}{\partial x}(x^2) - 0$$

$$= 3x^2 + 6y \cdot (2x)$$

$$= 3x^2 + 12xy.$$

$$f(x, y) = x^3 + 6x^2y - y^4$$

P.D w.r.to y

$$\frac{\partial f}{\partial y} = 0 + 6x^2 \frac{\partial}{\partial y}(y) - \frac{\partial}{\partial y}(y^4)$$

$$= 6x^2 \cdot (1) - 4y^3$$

$$= 6x^2 - 4y^3$$

③ $f(x, y) = x^5 + 3x^2y^3 - 6y^2$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Given that,

$$f(x, y) = x^5 + 3x^2y^3 - 6y^2$$

P.D w.r.to x

$$\frac{\partial f}{\partial x} = 5x^4 + 3y^3 \cdot \frac{\partial}{\partial x}(x^2) - 0$$

$$\frac{\partial f}{\partial x} = 5x^4 + 3y^3 \cdot (2x)$$

$$= 5x^4 + 6xy^3$$

$$f(x, y) = x^5 + 3x^2y^3 - 6y^2$$

P.D wrto y

$$\frac{\partial f}{\partial y} = 0 + 3x^2 \cdot \frac{\partial}{\partial y}(y^3) - 6 \cdot \frac{\partial}{\partial y}(y^2)$$

$$= 3x^2 \cdot (3y^2) - 6(2y)$$

$$= 9x^2y^2 - 12y$$

⇒ If $f = \sin(xy)$. find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$f = \sin(xy)$$

P.D wrto x

$$\frac{\partial f}{\partial x} = \cos(xy) \cdot \frac{\partial}{\partial x}(xy)$$

$$= \cos(xy) \cdot y \cdot \frac{\partial}{\partial x}(x)$$

$$= \cos(xy) \cdot y(1)$$

$$\frac{\partial f}{\partial x} = y \cos(xy)$$

$$f = \sin(xy)$$

PD wrto y

$$\frac{\partial f}{\partial y} = \cos(xy) \cdot \frac{\partial}{\partial y}(xy)$$

$$= \cos(xy) \cdot x \cdot \frac{\partial}{\partial y}(y)$$

$$= \cos(xy) \cdot x \cdot (1)$$

$$= \cos(xy) \cdot x$$

$$\frac{\partial f}{\partial y} = x \cos(xy)$$

Second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

Notes

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

If $f = x^2 y^3$. find (i) $\frac{\partial^2 f}{\partial x^2}$

(ii) $\frac{\partial^2 f}{\partial y^2}$

(iii) $\frac{\partial^2 f}{\partial x \partial y}$

(iv) $\frac{\partial^2 f}{\partial y \partial x}$

Sol

Given that,

(i) $f = x^2 y^3$

P.D wrto x

$$\frac{\partial f}{\partial x} = y^3 \cdot \frac{\partial}{\partial x} (x^2)$$

$$= y^3 \cdot 2x$$

$$\frac{\partial f}{\partial x} = 2xy^3$$

P.D wrto x

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2y^3 \frac{\partial}{\partial x} (x)$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 (1)$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3$$

(ii) $f = x^2 y^3$

P.D wrto y

$$\frac{\partial f}{\partial y} = x^2 \cdot \frac{\partial}{\partial y} (y^3)$$

$$= x^2 \cdot (3y^2)$$

$$\frac{\partial f}{\partial y} = 3x^2 y^2$$

P.D wrto y

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = 3x^2 \frac{\partial}{\partial y} (y^2)$$

$$\frac{\partial^2 f}{\partial y^2} = 3x^2 \cdot 2y$$

$$\frac{\partial^2 f}{\partial y^2} = 6x^2 y$$

$$(iii) f = x^2 y^3$$

p.d w.r.to y

$$\frac{\partial f}{\partial y} = x^2 \cdot \frac{\partial}{\partial y} (y^3)$$

$$= x^2 \cdot 3y^2$$

$$\frac{\partial f}{\partial y} = 3x^2 y^2$$

p.d w.r.to x

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = 3y^2 \cdot \frac{\partial}{\partial x} (x^2)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3y^2 \cdot (2x)$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

$$(iv) f = x^2 y^3$$

p.d w.r.to x

$$\frac{\partial f}{\partial x} = y^3 \cdot \frac{\partial}{\partial x} (x^2)$$

$$= y^3 \cdot (2x)$$

$$\frac{\partial f}{\partial x} = 2xy^3$$

p.d w.r.to y

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 2x \cdot \frac{\partial}{\partial y} (y^3)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x \cdot (3y^2)$$

$$\therefore \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

NOTE

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

① If $U = x^2 + y^2 + z^2$ then prove that

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = 2U$$

Given,

$$U = x^2 + y^2 + z^2$$

PD w.r.to x

$$\frac{\partial U}{\partial x} = 2x + 0 + 0$$

$$\frac{\partial U}{\partial x} = 2x$$

b.s multiply with x

$$x \cdot \frac{\partial U}{\partial x} = 2x \cdot x$$

$$x \cdot \frac{\partial U}{\partial x} = 2x^2 \text{ --- (1)}$$

E.T

$$U = x^2 + y^2 + z^2$$

PD w.r.to y

$$\frac{\partial U}{\partial y} = 0 + 2y + 0$$

$$\frac{\partial U}{\partial y} = 2y$$

b.s multiple with y

$$y \cdot \frac{\partial U}{\partial y} = 2y^2 \text{ --- (2)}$$

E.T,

$$U = x^2 + y^2 + z^2$$

PD w.r.to z

$$\frac{\partial U}{\partial z} = 0 + 0 + 2z$$

$$\frac{\partial U}{\partial z} = 2z$$

B.S multiply with z

$$z \cdot \frac{\partial U}{\partial z} = 2z^2 \text{ --- (3)}$$

Adding eq (1), (2) & (3)

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z}$$

$$= 2x^2 + 2y^2 + 2z^2$$

$$= 2(x^2 + y^2 + z^2)$$

$$= 2U$$

Hence proved

② If $U = \log(\tan x + \tan y + \tan z)$: Then show that

$$(\sin 2x) \frac{\partial U}{\partial x} + (\sin 2y) \frac{\partial U}{\partial y} + (\sin 2z) \frac{\partial U}{\partial z} = 2$$

Given,

$$U = \log(\tan x + \tan y + \tan z)$$

p.p. wrt to x

$$\frac{\partial U}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \frac{\partial}{\partial x} (\tan x + \tan y + \tan z)$$

$$\frac{\partial U}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot (\sec^2 x + 0 + 0)$$

$$\frac{\partial U}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

b.s multiply with $\sin 2x$

$$\sin 2x \cdot \frac{\partial U}{\partial x} = \frac{\sin 2x \cdot \sec^2 x}{\tan x + \tan y + \tan z}$$

$$(\because \sin x = 2 \sin x \cos x)$$

$$= 2 \sin x \cos x \cdot \frac{\frac{1}{\cos^2 x}}{\tan x + \tan y + \tan z}$$

$$= \frac{2 \frac{\sin x}{\cos x}}{\tan x + \tan y + \tan z}$$

$$\sin 2x \cdot \frac{\partial u}{\partial x} = \frac{2 \tan x}{\tan x + \tan y + \tan z} \quad \text{--- (1)}$$

similarly,

$$(\sin 2y) \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z} \quad \text{--- (2)}$$

$$(\sin 2z) \frac{\partial u}{\partial z} = \frac{2 \tan z}{\tan x + \tan y + \tan z} \quad \text{--- (3)}$$

Adding eq (1) (2) (3)

$$\begin{aligned} & (\sin 2x) \frac{\partial u}{\partial x} + (\sin 2y) \frac{\partial u}{\partial y} + (\sin 2z) \frac{\partial u}{\partial z} \\ &= \frac{2 \tan x}{\tan x + \tan y + \tan z} + \frac{2 \tan y}{\tan x + \tan y + \tan z} + \frac{2 \tan z}{\tan x + \tan y + \tan z} \\ &= \frac{2 \tan x + 2 \tan y + 2 \tan z}{\tan x + \tan y + \tan z} \\ &= \frac{2 (\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} \\ &= 2 \end{aligned}$$

$$(\sin 2x) \frac{\partial u}{\partial x} + (\sin 2y) \frac{\partial u}{\partial y} + (\sin 2z) \frac{\partial u}{\partial z} = 2$$

Hence proved //

3. If $U = x^2(y-z) + y^2(z-x) + z^2(x-y)$ Then

$$\text{P.T. } \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$$

sol G.T

$$U = x^2(y-z) + y^2(z-x) + z^2(x-y)$$

P.D wrto x

$$\frac{\partial U}{\partial x} = (y-z) \frac{\partial}{\partial x}(x^2) + y^2 \frac{\partial}{\partial x}(z-x) + z^2 \frac{\partial}{\partial x}(x-y)$$

$$= (y-z)(2x) + y^2(0-1) + z^2(1-0)$$

$$= 2xy - 2xz + (-y^2) + z^2$$

$$\frac{\partial U}{\partial x} = 2xy - 2xz - y^2 + z^2 \text{ --- ①}$$

$$U = x^2(y-z) + y^2(z-x) + z^2(x-y)$$

P.D wrto y

$$\frac{\partial U}{\partial y} = x^2 \frac{\partial}{\partial y}(y-z) + (z-x) \frac{\partial}{\partial y}(y^2) + z^2 \frac{\partial}{\partial y}(x-y)$$

$$= x^2(1-0) + (z-x)(2y) + z^2(0-1)$$

$$= x^2(1) + 2xyz - 2xy + z^2(-1)$$

$$\frac{\partial U}{\partial y} = x^2 + 2yz - 2xy - z^2 \text{ --- ②}$$

$$v = x^2(y-z) + y^2(z-x) + z^2(x-y)$$

p.d w.r.to z

$$\frac{\partial v}{\partial z} = x^2 \cdot \frac{\partial}{\partial z}(y-z) + y^2 \cdot \frac{\partial}{\partial z}(z-x) + (x-y) \frac{\partial}{\partial z}(z^2)$$

$$= x^2(0-1) + y^2(1-0) + (x-y)(2z)$$

$$= -x^2 + y^2 + 2xz - 2yz$$

$$\frac{\partial v}{\partial z} = -x^2 + y^2 + 2xz - 2yz \quad \text{--- (3)}$$

Adding eq (1), (2), (3)

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = \cancel{2xy} - \cancel{2xz} - \cancel{y^2} + \cancel{z^2} + \cancel{x^2} + \cancel{2yz} - \cancel{2xy} - \cancel{z^2} - \cancel{x^2} + \cancel{y^2} + \cancel{2xz} - \cancel{2yz}$$

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = 0$$

Hence proved.

4. If $z = f(x+ay) + \psi(x-ay)$ Then prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \cdot \frac{\partial^2 z}{\partial x^2}$$

G.T,

$$z = f(x+ay) + \psi(x-ay)$$

p.d w.r.to y

$$\frac{\partial z}{\partial y} = f'(x+ay) \cdot \frac{\partial}{\partial y}(x+ay) + \psi'(x-ay) \cdot \frac{\partial}{\partial y}(x-ay)$$

$$\frac{\partial z}{\partial y} = f'(x+ay) \cdot (0+a(1)) + \psi'(x-ay) \cdot (0-a(1))$$

$$\frac{\partial z}{\partial y} = a f'(x+ay) - a \psi'(x-ay)$$

again P.D wrts y

$$\left(\frac{\partial^2 z}{\partial y^2} = a f''(x+ay) - a \psi''(x-ay) \right) \times$$

$$\frac{\partial^2 z}{\partial y^2} = a f''(x+ay) \cdot \frac{\partial}{\partial y}(x+ay) - a \psi''(x-ay) \cdot \frac{\partial}{\partial y}(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = a f''(x+ay) (0+a(1)) - a \psi''(x-ay) (0-a(1))$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 \psi''(x-ay) \quad \text{--- (1)}$$

$$z = f(x+ay) + \psi(x-ay)$$

P.D wrts x

$$\frac{\partial z}{\partial x} = f'(x+ay) \cdot \frac{\partial}{\partial x}(x+ay) + \psi'(x-ay) \cdot \frac{\partial}{\partial x}(x-ay)$$

$$\frac{\partial z}{\partial x} = f'(x+ay) \cdot (1+0) + \psi'(x-ay) \cdot (1-0)$$

$$\frac{\partial z}{\partial x} = f'(x+ay) + \psi'(x-ay)$$

Again P.D w.r.to x

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) \frac{\partial}{\partial x}(x+ay) + \psi''(x-ay) \cdot \frac{\partial}{\partial x}(x-ay)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) \cdot (1+0) + \psi''(x-ay) \cdot (1-0)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \psi''(x-ay) \quad \text{--- (1)}$$

both sides multiply with a^2 .

$$a^2 \cdot \frac{\partial^2 z}{\partial x^2} = a^2 f''(x+ay) + a^2 \psi''(x-ay) \quad \text{--- (2)}$$

Substituting from eq (1) and (2)

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Hence proved.

5. If $U = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ Then S-T $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = 20$

G-T, $U = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

P.D w.r.to x

$$\frac{\partial U}{\partial x} = \frac{1}{y} \cdot \frac{\partial}{\partial x}(x) + 0 + z \cdot \frac{\partial}{\partial x}\left(\frac{1}{x}\right)$$

$$\frac{\partial U}{\partial x} = \frac{1}{y} (1) + z \left(-\frac{1}{x^2}\right)$$

$$\frac{\partial v}{\partial x} = \frac{1}{y} - \frac{z}{x^2}$$

both sides multi with x

$$x \cdot \frac{\partial v}{\partial x} = \frac{x}{y} - \frac{xz}{x^2}$$

$$x \cdot \frac{\partial v}{\partial x} = \frac{x}{y} - \frac{z}{x} \quad \text{--- (1)}$$

$$v = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

P.D wrto y

$$\frac{\partial v}{\partial y} = x \cdot \frac{\partial}{\partial y} \left(\frac{1}{y} \right) + \frac{1}{z} \cdot \frac{\partial}{\partial y} (y) + 0$$

$$\frac{\partial v}{\partial y} = x \cdot \left(-\frac{1}{y^2} \right) + \frac{1}{z} (1) + 0$$

$$\frac{\partial v}{\partial y} = -\frac{x}{y^2} + \frac{1}{z}$$

both sides multi with y .

$$y \cdot \frac{\partial v}{\partial y} = \frac{-xy}{y^2} + \frac{y}{z}$$

$$y \cdot \frac{\partial v}{\partial y} = \frac{-x}{y} + \frac{y}{z} \quad \text{--- (2)}$$

$$v = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

P.D wrto z

$$\frac{\partial v}{\partial z} = 0 + y \cdot \frac{\partial}{\partial z} \left(\frac{1}{z} \right) + \frac{1}{x} \cdot \frac{\partial}{\partial z} (z)$$

$$\frac{\partial v}{\partial z} = 0 + y \cdot \left(-\frac{1}{z^2} \right) + \frac{1}{x} \cdot (1)$$

$$\frac{\partial v}{\partial z} = -\frac{y}{z^2} + \frac{1}{x}$$

both sides multi with z .

$$z \cdot \frac{\partial v}{\partial z} = -\frac{zy}{z^2} + \frac{z}{x}$$

$$z \cdot \frac{\partial v}{\partial z} = -\frac{y}{z} + \frac{z}{x} \quad \text{--- (3)}$$

Adding eq (1), (2) & (3)

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + z \cdot \frac{\partial v}{\partial z} = \cancel{\frac{x}{y}} - \cancel{\frac{y}{x}} - \cancel{\frac{x}{y}} + \cancel{\frac{y}{z}} - \cancel{\frac{y}{z}} + \cancel{\frac{z}{x}}$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + z \cdot \frac{\partial v}{\partial z} = 0$$

Hence proved.

6. If $U = f(x^2 + y^2)$ - Then s.t $y \cdot \frac{\partial U}{\partial x} = x \cdot \frac{\partial U}{\partial y}$

$$U = f(x^2 + y^2)$$

P.D wrto x

$$\frac{\partial U}{\partial x} = f'(x^2 + y^2) \cdot \frac{\partial}{\partial x}(x^2 + y^2)$$

$$\frac{\partial U}{\partial x} = f'(x^2 + y^2) \cdot (2x + 0)$$

$$\frac{\partial U}{\partial x} = 2x \cdot f'(x^2 + y^2) \text{ --- (1)}$$

$$U = f(x^2 + y^2)$$

P.D wrto y

$$\frac{\partial U}{\partial y} = f'(x^2 + y^2) \cdot \frac{\partial}{\partial y}(x^2 + y^2)$$

$$\frac{\partial U}{\partial y} = f'(x^2 + y^2) \cdot (0 + 2y)$$

$$\frac{\partial U}{\partial y} = 2y \cdot f'(x^2 + y^2) \text{ --- (2)}$$

eq(1)/eq(2)

$$\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} = \frac{2x \cdot f'(x^2 + y^2)}{2y \cdot f'(x^2 + y^2)}$$

$$y \cdot \frac{\partial U}{\partial x} = x \cdot \frac{\partial U}{\partial y}$$

Hence proved //

Example:

Q If $U = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$ and $l^2 + m^2 + n^2 = 1$

Then ST. $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$

sol

G.T., $U = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$

PD wrto x

$$\frac{\partial U}{\partial x} = 3 \cdot \frac{\partial}{\partial x} (lx + my + nz)^2 - \frac{\partial}{\partial x} (x^2 + y^2 + z^2)$$

$$\frac{\partial U}{\partial x} = 3 \cdot [l(1) + 0 + 0]$$

$$\frac{\partial U}{\partial x} = 3 \cdot 2(lx + my + nz)^{2-1} \cdot \frac{\partial}{\partial x} (lx + my + nz) - (2x + 0 + 0)$$

$$\frac{\partial U}{\partial x} = 6(lx + my + nz)(l + 0 + 0) - 2x$$

$$\frac{\partial U}{\partial x} = 6l(lx + my + nz) - 2x$$

Again PD wrto x

$$\frac{\partial^2 U}{\partial x^2} = 6l \frac{\partial}{\partial x} (lx + my + nz) - \frac{\partial}{\partial x} (2x)$$

$$\frac{\partial^2 U}{\partial x^2} = 6l[l + 0 + 0] - 2(1)$$

$$\frac{\partial^2 U}{\partial x^2} = 6l^2 - 2 \text{ --- ①}$$

$$U = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$$

P.D wrto y

$$\frac{\partial U}{\partial y} = 3 \cdot \frac{\partial}{\partial y} (lx + my + nz)^2 - \frac{\partial}{\partial y} (x^2 + y^2 + z^2)$$

$$\frac{\partial U}{\partial y} = 3 \cdot 2(lx + my + nz)^{2-1} \cdot \frac{\partial}{\partial y} (lx + my + nz) - (0 + 2y + 0)$$

$$\frac{\partial U}{\partial y} = 6(lx + my + nz) \cdot (0 + m + 0) - 2y$$

$$\frac{\partial U}{\partial y} = 6m(lx + my + nz) - 2y$$

again P.D wrto y

$$\frac{\partial^2 U}{\partial y^2} = 6m \frac{\partial}{\partial y} (lx + my + nz) - \frac{\partial}{\partial y} (2y)$$

$$\frac{\partial^2 U}{\partial y^2} = 6m(0 + m + 0) - 2$$

$$\frac{\partial^2 U}{\partial y^2} = 6m^2 - 2 \quad \text{--- (2)}$$

$$U = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$$

P.D wrto z

$$\frac{\partial U}{\partial z} = 3 \cdot \frac{\partial}{\partial z} (lx + my + nz)^2 - \frac{\partial}{\partial z} (x^2 + y^2 + z^2)$$

$$\frac{\partial v}{\partial z} = 3 \cdot 2(lx + my + nz)^{2-1} \frac{\partial}{\partial z}(lx + my + nz) - (0 + 0 + 2z)$$

$$\frac{\partial v}{\partial z} = 6(lx + my + nz)(0 + 0 + 1) - 2z$$

$$\frac{\partial v}{\partial z} = 6z(lx + my + nz) - 2z$$

Again p.v w.r to z

$$\frac{\partial^2 v}{\partial z^2} = 6z \cdot \frac{\partial}{\partial z}(lx + my + nz) - \frac{\partial}{\partial z}(2z)$$

$$\frac{\partial^2 v}{\partial z^2} = 6z(0 + 0 + 1) - 2$$

$$\frac{\partial^2 v}{\partial z^2} = 6z - 2 \quad \text{--- (3)}$$

Adding ①, ②, ③

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 6l^2 - 2 + 6m^2 - 2 + 6n^2 - 2$$

$$= 6l^2 + 6m^2 + 6n^2 - 6$$

$$= 6[l^2 + m^2 + n^2 - 1]$$

$$= 6[l^2 + m^2 + n^2] - 6$$

$$= 6(1) - 6 \quad [\because l^2 + m^2 + n^2 = 1]$$

$$= 0$$

Hence proved

Another method

$$\text{Similarly } \frac{\partial^2 v}{\partial y^2} = 6m^2 - 2 \quad \text{--- (2)}, \quad \frac{\partial^2 v}{\partial z^2} = 6n^2 - 2 \quad \text{--- (3)}$$

Adding ① ② ③

Example:

Q If $U = \frac{1}{\sqrt{x^2+y^2+z^2}}$ Then ST

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

Sol GT,

$$U = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$U = \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$U = (x^2+y^2+z^2)^{-1/2}$$

P.D wrto x

$$\frac{\partial U}{\partial x} = -\frac{1}{2} (x^2+y^2+z^2)^{-1/2-1} \cdot \frac{\partial}{\partial x} (x^2+y^2+z^2)$$

$$\frac{\partial U}{\partial x} = -\frac{1}{2} (x^2+y^2+z^2)^{-3/2} \cdot (2x+0+0)$$

$$\frac{\partial U}{\partial x} = -\frac{1}{2} 2x (x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial U}{\partial x} = -x (x^2+y^2+z^2)^{-3/2}$$

Again p.d wrto x

$$\frac{\partial^2 U}{\partial x^2} = (-x) \frac{\partial}{\partial x} (x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2} \cdot \frac{\partial}{\partial x} (-x)$$

$$\frac{\partial^2 U}{\partial x^2} = (-x)(-3x)(x^2+y^2+z^2)^{-3/2} + \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2} \cdot (-1)$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{3x}{2}(x^2+y^2+z^2)^{-5/2} \cdot (2x+0+0) + (x^2+y^2+z^2)^{-3/2} \cdot (-1)$$

$$\frac{\partial^2 U}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (1)}$$

Similarly,

$$\frac{\partial^2 U}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (2)}$$

$$\frac{\partial^2 U}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (3)}$$

Adding eq (1), (2), (3)

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-5/2+1} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

Hence proved //

2019 (4M)

If $f(x, y) = y \cos(xy)$ then evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Given $f(x, y) = y \cos(xy)$

$$f = y \cos(xy)$$

P.D wrto x

$$\frac{\partial f}{\partial x} = y \cdot \frac{\partial}{\partial x} (\cos xy)$$

$$\frac{\partial f}{\partial x} = y \cdot (-\sin(xy)) \cdot \frac{\partial}{\partial x} (xy)$$

$$\frac{\partial f}{\partial x} = y \cdot (-\sin(xy)) \cdot y \cdot \frac{\partial}{\partial x} (x)$$

$$\frac{\partial f}{\partial x} = -y \sin(xy) \cdot y (1)$$

$$\frac{\partial f}{\partial x} = -y^2 \sin(xy)$$

$$f = y \cos(xy)$$

P.D wrto y

$$\frac{\partial f}{\partial y} = y \cdot \frac{\partial}{\partial y} (\cos(xy)) + \cos(xy) \cdot \frac{\partial}{\partial y} (y)$$

$$\frac{\partial f}{\partial y} = y \cdot (-\sin(xy)) \cdot \frac{\partial}{\partial x} (xy) + \cos(xy) \cdot 1$$

$$\frac{\partial f}{\partial y} = y(-\sin(xy)) \cdot x(1) + \cos(xy)$$

$$\frac{\partial f}{\partial y} = -xy \sin(xy) + \cos(xy)$$

$$\frac{\partial f}{\partial y} = -xy \sin(xy) + \cos(xy)$$

$$u_x = \frac{\partial u}{\partial x}$$

Q If $U = x + y + z$ $V = x^2 + y^2 + z^2$ $W = x^3 + y^3 + z^3 - 3xyz$

Then prove that $\begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix} = 0$ $\left[\begin{array}{l} U_x = \frac{\partial U}{\partial x} \\ V_y = \frac{\partial V}{\partial y} \\ W_z = \frac{\partial W}{\partial z} \end{array} \right]$

Given, $U = x + y + z$

$$V = x^2 + y^2 + z^2$$

$$W = x^3 + y^3 + z^3 - 3xyz$$

$$U = x + y + z$$

$$V = x^2 + y^2 + z^2$$

$$W = x^3 + y^3 + z^3 - 3xyz$$

$$U_x = 1 + 0 + 0$$

$$V_x = 2x + 0 + 0$$

$$W_x = 3x^2 + 0 + 0 - 3yz(1)$$

$$\boxed{U_x = 1}$$

$$\boxed{V_x = 2x}$$

$$\boxed{W_x = 3x^2 - 3yz}$$

$$U_y = 0 + 1 + 0$$

$$V_y = 0 + 2y + 0$$

$$W_y = 0 + 3y^2 + 0 - 3xz(1)$$

$$\boxed{U_y = 1}$$

$$\boxed{V_y = 2y}$$

$$\boxed{W_y = 3y^2 - 3xz}$$

$$U_z = 0 + 0 + 1$$

$$V_z = 0 + 0 + 2z$$

$$W_z = 0 + 0 + 3z^2 - 3xy(1)$$

$$\boxed{U_z = 1}$$

$$\boxed{V_z = 2z}$$

$$\boxed{W_z = 3z^2 - 3xy}$$

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2y & 2z \\ 3y^2-3xz & 3z^2-3xy \end{vmatrix} - 1 \begin{vmatrix} 2x & 2z \\ 3x^2-3yz & 3z^2-3xy \end{vmatrix} + 1 \begin{vmatrix} 2x & 2y \\ 3x^2-3yz & 3y^2-3xz \end{vmatrix}$$

$$= 1 [2y(3z^2-3xy) - 2z(3y^2-3xz)] - 1 [2x(3z^2-3xy) - 2z(3x^2-3yz)] + 1 [2x(3y^2-3xz) - 2y(3x^2-3yz)]$$

$$= 1 [6yz^2 - 6xy^2 - 6y^2z + 6xz^2] - 1 [6xz^2 - 6x^2y - 6x^2z + 6yz^2] + 6xy^2 - 6x^2z - 6x^2y + 6y^2z$$

$$= 6yz^2 - 6xy^2 - 6y^2z + 6xz^2 - 6xz^2 + 6x^2y + 6x^2z - 6yz^2 + 6xy^2 - 6x^2z - 6x^2y + 6y^2z$$

$$= 0$$

$$\therefore \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$$

Hence proved.

Q. If $U = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$ Show that $U_x + U_y + U_z = 0$

$$U = x^2 \begin{vmatrix} y & z \\ 1 & 1 \end{vmatrix} - y^2 \begin{vmatrix} x & z \\ 1 & 1 \end{vmatrix} + z^2 \begin{vmatrix} x & y \\ 1 & 1 \end{vmatrix}$$

$$= x^2[y-z] - y^2[x-z] + z^2[x-y]$$

$$\times \left\{ \begin{aligned} &= x^2y - x^2z - xy^2 + y^2z + z^2x - yz^2 \\ &= x^2y - x^2z - xy^2 + y^2z + z^2x - yz^2 \end{aligned} \right\} \times$$

$$= x^2[y-z] - y^2[x-z] + z^2[x-y]$$

$$U = x^2[y-z] + y^2[z-x] + z^2[x-y]$$

P.D wrto x

$$\frac{\partial U}{\partial x} = (y-z) \cdot \frac{\partial}{\partial x}(x^2) + y^2 \frac{\partial}{\partial x}[z-x] + z^2 \frac{\partial}{\partial x}(x-y)$$

$$\frac{\partial U}{\partial x} = (y-z)(2x) + y^2(0-1) + z^2(1-0)$$

$$\frac{\partial U}{\partial x} = 2x(y-z) - y^2 + z^2$$

$$\frac{\partial U}{\partial x} = 2xy - 2xz - y^2 + z^2 \text{ --- ①}$$

$$U = x^2[y-z] + y^2[z-x] + z^2[x-y]$$

P.D wrto y

$$\frac{\partial u}{\partial y} = x^2 \cdot \frac{\partial}{\partial y}(y-z) + (z-x) \frac{\partial}{\partial y}(y^2) + z^2 \frac{\partial}{\partial y}(x-y)$$

$$\frac{\partial u}{\partial y} = x^2 \cdot (1-0) + (z-x)(2y) + z^2(0-1)$$

$$\frac{\partial u}{\partial y} = x^2 + 2yz - 2xy - z^2 \quad \text{--- (2)}$$

$$u = x^2(y-z) + y^2(z-x) + z^2(x-y)$$

P.D w.r.to z

$$\frac{\partial u}{\partial z} = x^2 \cdot \frac{\partial}{\partial z}(y-z) + y^2 \cdot \frac{\partial}{\partial z}(z-x) + (x-y) \frac{\partial}{\partial z}(z^2)$$

$$\frac{\partial u}{\partial z} = x^2(0-1) + y^2(1-0) + (x-y)(2z)$$

$$\frac{\partial u}{\partial z} = -x^2 + y^2 + 2xz - 2yz \quad \text{--- (3)}$$

Adding (1), (2) (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2xy - 2xz - y^2 + z^2 + x^2 + 2yz - 2xy - z^2 - x^2 + y^2 + 2xz - 2yz$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$$\text{i.e., } u_x + u_y + u_z = 0 //$$

Hence proved.

NOTE:

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

* (Igm)

Example:

If $U = \log(x^3 + y^3 + z^3 - 3xyz)$ then show that

$$(i) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 U = \frac{-9}{(x+y+z)^2} \Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) U$$

$$(ii) \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{-3}{(x+y+z)^2} = \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} \right]$$

Given,

$$U = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\left[\frac{d}{dx}(\log x) = \frac{1}{x} \right]$$

P.D w.r.to x

$$\frac{\partial U}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \cdot \frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial U}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} [3x^2 + 0 + 0 - 3yz(1)]$$

$$\frac{\partial U}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} [3x^2 - 3yz]$$

$$\frac{\partial U}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \quad \text{--- (1)}$$

Similarly,

$$\frac{\partial v}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \quad \text{--- (2)}$$

$$\frac{\partial v}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \quad \text{--- (3)}$$

Adding eq (1), (2), (3)

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3[x^2 - yz + y^2 - xz + z^2 - xy]}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3[x^2 + y^2 + z^2 - xz - yz - xy]}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = \frac{3}{x+y+z} \quad \text{--- (4)}$$

$$(i) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 \cdot U$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot U$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{3}{x+y+z} \right) \quad [\text{from eq (4)}]$$

$$= \frac{\partial}{\partial x} \left(\frac{3}{x+y+z} \right) + \frac{\partial}{\partial y} \left(\frac{3}{x+y+z} \right) + \frac{\partial}{\partial z} \left(\frac{3}{x+y+z} \right) \quad \left[\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2} \right]$$

$$= 3 \cdot \frac{\partial}{\partial x} \left(\frac{1}{x+y+z} \right) + 3 \cdot \frac{\partial}{\partial y} \left(\frac{1}{x+y+z} \right) + 3 \cdot \frac{\partial}{\partial z} \left(\frac{1}{x+y+z} \right)$$

$$= 3 \cdot \frac{-1}{(x+y+z)^2} \cdot \frac{\partial}{\partial x} (x+y+z) + 3 \cdot \frac{-1}{(x+y+z)^2} \cdot \frac{\partial}{\partial y} (x+y+z) + 3 \cdot \frac{-1}{(x+y+z)^2} \cdot \frac{\partial}{\partial z} (x+y+z)$$

$$= 3 \cdot \frac{-1}{(x+y+z)^2} (1+0+0) + 3 \cdot \frac{-1}{(x+y+z)^2} (0+1+0) + 3 \cdot \frac{-1}{(x+y+z)^2} (0+0+1)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-3-3-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

$$\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 \cdot u = \frac{-9}{(x+y+z)^2}$$

Hence proved.

(ii) from eq(1)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{d}{dx}(u) - u \frac{d}{dx}(v)}{v^2}$$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

PD wrto x on B.S

$$\frac{\partial^2 u}{\partial x^2} = \frac{x^3 + y^3 + z^3 - 3xyz \cdot \frac{\partial}{\partial x}(3x^2 - 3yz) - (x^3 + y^3 + z^3 - 3xyz) \frac{\partial}{\partial x}(3x^2 - 3yz)}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$= \frac{x^3 + y^3 + z^3 - 3xyz \cdot [6x - 0] - (x^3 + y^3 + z^3 - 3xyz) [X]}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{x^3 + y^3 + z^3 - 3xyz \cdot \frac{\partial}{\partial x}(3x^2 - 3yz) - (3x^2 - 3yz) \frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz)}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$= \frac{x^3 + y^3 + z^3 - 3xyz (6x - 0) - (3x^2 - 3yz)(3x^2 + 0 + 0 - 3yz)}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$= \frac{(6x^4 + 6xy^3 + 6xz^3 - 18x^2yz) - (9x^4 - 9x^2yz - 9x^2yz + 9y^2z^2)}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{6x^4 + 6xy^3 + 6xz^3 - 18x^2yz - 9x^4 + 9x^2yz + 9x^2yz - 9y^2z^2}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-3x^4 + 6xy^3 + 6xz^3 - 9y^2z^2}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{-3(x^4 - 2xy^3 - 2xz^3 + 3y^2z^2)}{(x^3 + y^3 + z^3 - 3xyz)^2} \quad \text{--- (5)}$$

Similarly,

$$\frac{\partial^2 U}{\partial y^2} = \frac{-3(y^4 - 2yx^3 - 2yz^3 + 3x^2z^2)}{(x^3 + y^3 + z^3 - 3xyz)^2} \quad \text{--- (6)}$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{-3(z^4 - 2zx^3 - 2zy^3 + 3x^2y^2)}{(x^3 + y^3 + z^3 - 3xyz)^2} \quad \text{--- (7)}$$

Adding eq (5) (6) (7)

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{-3[x^4 - 2xy^3 - 2xz^3 + 3y^2z^2 + y^4 - 2yx^3 - 2yz^3 + 3x^2z^2 + z^4 - 2zx^3 - 2zy^3 + 3x^2y^2]}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$= \frac{-3[x^2 + y^2 + z^2 - xy - yz - zx]^2}{[(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)]^2}$$

$$= \frac{-3[x^2 + y^2 + z^2 - xy - yz - zx]^2}{(x+y+z)^2 (x^2 + y^2 + z^2 - xy - yz - zx)^2}$$

$$= \frac{-3}{(x+y+z)^2}$$

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{-3}{(x+y+z)^2}$$

$$\text{Hence proved.}$$

2019 (um)

If $f(x, y) = \frac{xy}{x^2 + y^2}$. Then evaluate $\frac{\partial^2 f}{\partial x \partial y}$

$$f = \frac{xy}{x^2 + y^2}$$

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{d}{dx}(u) - u \frac{d}{dx}(v)}{v^2}$$

P.D wrto y

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2) \frac{\partial}{\partial y}(xy) - xy \cdot \frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2) x \frac{\partial}{\partial y}(y) - xy(0 + 2y)}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2) x (1) - 2xy^2}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2) x (1) - 2xy^2}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^3 + xy^2 - 2xy^2}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^3 - xy^2}{(x^2 + y^2)^2}$$

Again PD wrto x

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{(x^2 + y^2)^2 \cdot \frac{\partial}{\partial x}(x^3 - xy^2) - (x^3 - xy^2) \frac{\partial}{\partial x}(x^2 + y^2)^2}{((x^2 + y^2)^2)^2}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{(x^2+y^2)^2(3x^2-y^2) - (x^3-xy^2) \cdot 2(x^2+y^2) \cdot \frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{(x^2+y^2)^2(3x^2-y^2) - (x^3-xy^2) \cdot 2(x^2+y^2) \cdot (2x)}{(x^2+y^2)^4}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{(x^2+y^2)^2 \left[(3x^2-y^2)(x^2+y^2) - 4(x^3-xy^2)(x) \right]}{(x^2+y^2)^4}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{(x^2+y^2)(3x^2-y^2) - 4(x^3-xy^2)x}{(x^2+y^2)^3}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{3x^4 - x^2y^2 + 3x^2y^2 - y^4 - 4x^4 + 4x^2y^2}{(x^2+y^2)^3}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{-x^4 - y^4 + 6x^2y^2}{(x^2+y^2)^3}$$

$$\frac{d}{dx}(\tan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan^{-1} f(x)) = \frac{1}{1+(f(x))^2} \cdot \frac{d}{dx}(f(x))$$

$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{1}{y} \cdot \frac{d}{dx}(x) = \frac{1}{y} (1) = \frac{1}{y}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = y \cdot \frac{d}{dx}\left(\frac{1}{x}\right) = y \cdot \left(-\frac{1}{x^2}\right) = -\frac{y}{x^2}$$

Example:

$$\text{If } U = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

$$\text{Then p.t. } \frac{\partial^2 U}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

Sol Given that,

$$U = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

p.d wrto y.

$$\frac{\partial U}{\partial y} = x^2 \cdot \frac{\partial}{\partial y} \left[\tan^{-1}\left(\frac{y}{x}\right) \right] - \left[y^2 \cdot \frac{\partial}{\partial y} \left[\tan^{-1}\left(\frac{x}{y}\right) \right] + \tan^{-1}\left(\frac{x}{y}\right) \cdot \frac{\partial}{\partial y}(y^2) \right]$$

$$\frac{\partial U}{\partial y} = x^2 \cdot \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) - \left[y^2 \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y \right]$$

$$\frac{\partial U}{\partial y} = x^2 \cdot \frac{1}{\frac{x^2 + y^2}{x^2}} \cdot \frac{1}{x} \cdot \frac{\partial}{\partial y}(y) - \left[y^2 \cdot \frac{1}{\frac{y^2 + x^2}{y^2}} \cdot x \cdot \frac{\partial}{\partial y}\left(\frac{1}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y \right]$$

$$\frac{\partial v}{\partial y} = x^2 \cdot \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} (1) - \left[y^2 \cdot \frac{y^2}{y^2+x^2} \cdot x \cdot \left[-\frac{1}{y^2} \right] + \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y \right]$$

$$\frac{\partial v}{\partial y} = \frac{x^4}{x^2+y^2} \cdot \frac{1}{x} - \left[\frac{y^4}{y^2+x^2} \cdot \left[-\frac{x}{y^2} \right] + \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y \right]$$

$$\frac{\partial v}{\partial y} = \frac{x^3}{x^2+y^2} \cdot \frac{1}{x} + \frac{xy^2}{y^2+x^2} + \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y$$

$$\frac{\partial v}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{y^2+x^2} - \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y$$

$$\frac{\partial v}{\partial y} = \frac{x^3 + xy^2}{x^2+y^2} - 2y \cdot \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial v}{\partial y} = \frac{x(x^2+y^2)}{x^2+y^2} - 2y \cdot \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial v}{\partial y} = x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

p.d wrts x

$$\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = 1 - 2y \cdot \frac{\partial}{\partial x} \left(\tan^{-1}\left(\frac{x}{y}\right) \right)$$

$$\frac{\partial^2 v}{\partial x \partial y} = 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{\partial}{\partial x} \left(\frac{x}{y} \right)$$

$$\frac{\partial^2 v}{\partial x \partial y} = 1 - 2y \cdot \frac{1}{\frac{y^2+x^2}{y^2}} \cdot \frac{1}{y} (1)$$

$$\frac{\partial^2 u}{\partial x \partial y} = 1 - 2y \cdot \frac{y^2}{y^2 + x^2} \cdot \frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 1 - \frac{2y^2}{y^2 + x^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{y^2 + x^2 - 2y^2}{y^2 + x^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

Hence proved.

If $U = \log \sqrt{x^2 + y^2 + z^2}$ Then P.T

$$(x^2 + y^2 + z^2) \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] = 1$$

Given,

$$U = \log \sqrt{x^2 + y^2 + z^2}$$

$$U = \log (x^2 + y^2 + z^2)^{1/2}$$

$$U = \frac{1}{2} \log (x^2 + y^2 + z^2)$$

P.D wrto x

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2 + z^2} \cdot \frac{\partial}{\partial x} (x^2 + y^2 + z^2)$$

$$\boxed{\log a^m = m \cdot \log a}$$

$$\frac{\partial v}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2+z^2} \cdot [2x+0+0]$$

$$\frac{\partial v}{\partial x} = \frac{1}{2} \cdot \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2+z^2}$$

p.p wrto x

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2+z^2) \frac{\partial}{\partial x}(x) - x \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2+z^2)(1) - x \cdot (2x+0+0)}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{x^2+y^2+z^2 - 2x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2} \quad \text{--- ①}$$

Similarly,

$$\frac{\partial^2 v}{\partial y^2} = \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2} \quad \text{--- ②}$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \quad \text{--- ③}$$

Adding eq (1) (2) (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2} + \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-x^2+y^2+z^2 + x^2-y^2+z^2 + x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{x^2+y^2+z^2} \quad (\text{or})$$

$$\therefore x^2+y^2+z^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] = 1$$

Hence proved. //

(or)

$$\text{LHS} = x^2+y^2+z^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

$$= x^2+y^2+z^2 \left[\frac{1}{x^2+y^2+z^2} \right]$$

$$= 1$$

$$= \text{RHS}$$

Hence proved //

Example
If $U = e^{xyz}$. Then S.T

$$\frac{\partial^3 U}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2) e^{xyz}$$

Given that

$$U = e^{xyz}$$

P.D wrto z

$$\frac{\partial U}{\partial z} = e^{xyz} \cdot \frac{\partial}{\partial z} (xyz)$$

$$\frac{\partial U}{\partial z} = e^{xyz} \cdot xy \cdot \frac{\partial}{\partial z} (z)$$

$$\frac{\partial U}{\partial z} = e^{xyz} \cdot xy (1)$$

$$\frac{\partial U}{\partial z} = e^{xyz} \cdot xy$$

Again P.D wrto y

$$\frac{\partial}{\partial y} \left[\frac{\partial U}{\partial z} \right] = \frac{\partial}{\partial y} (e^{xyz} \cdot xy)$$

$$\frac{\partial^2 U}{\partial y \partial z} = e^{xyz} \cdot \frac{\partial}{\partial y} (xy) + xy \cdot \frac{\partial}{\partial y} (e^{xyz})$$

$$\frac{\partial^2 U}{\partial y \partial z} = e^{xyz} \cdot x \cdot \frac{\partial}{\partial y} (y) + xy \cdot e^{xyz} \cdot \frac{\partial}{\partial y} (xyz)$$

$$\frac{\partial^2 U}{\partial y \partial z} = e^{xyz} \cdot x (1) + xy \cdot e^{xyz} \cdot xz \cdot \frac{\partial}{\partial y} (y)$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz} \cdot x + xy \cdot e^{xyz} \cdot xz(1)$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz} \cdot x + e^{xyz} \cdot x^2 y z$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz} (x + x^2 y z)$$

Again PD wrto x

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial y \partial z} \right) = \frac{\partial}{\partial x} [e^{xyz} (x + x^2 y z)]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} \frac{\partial}{\partial x} (x + x^2 y z) + (x + x^2 y z) \frac{\partial}{\partial x} (e^{xyz})$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} (1 + yz(2x)) + (x + x^2 y z) e^{xyz} \cdot \frac{\partial}{\partial x} (xyz)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} (1 + 2xyz) + (x + x^2 y z) e^{xyz} \cdot yz(1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} [(1 + 2xyz) + (x + x^2 y z) yz]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} [1 + 2xyz + xyz + x^2 y^2 z^2]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} [1 + 3xyz + x^2 y^2 z^2] //$$

Hence proved.

If $U = \log(x^2 + y^2 + z^2)$. Then prove that

$$x \cdot \frac{\partial^2 U}{\partial y \partial z} = y \cdot \frac{\partial^2 U}{\partial z \partial x} = z \cdot \frac{\partial^2 U}{\partial x \partial y}$$

Given that

$$U = \log(x^2 + y^2 + z^2)$$

$$\frac{\partial^2 U}{\partial y \partial z}$$

P.D wrto z

$$\frac{\partial U}{\partial z} = \frac{1}{x^2 + y^2 + z^2} \cdot \frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$\frac{\partial U}{\partial z} = \frac{1}{x^2 + y^2 + z^2} (0 + 0 + 2z)$$

$$\frac{\partial U}{\partial z} = \frac{2z}{x^2 + y^2 + z^2}$$

Again PD wrto y

$$\frac{\partial}{\partial y} \left(\frac{\partial U}{\partial z} \right) = 2z \cdot \frac{\partial}{\partial y} \left(\frac{1}{x^2 + y^2 + z^2} \right)$$

$$\frac{\partial^2 U}{\partial y \partial z} = 2z \cdot \frac{(-1)}{(x^2 + y^2 + z^2)^2} \cdot \frac{\partial}{\partial y}(x^2 + y^2 + z^2)$$

$$\frac{\partial^2 U}{\partial y \partial z} = \frac{-2z}{(x^2 + y^2 + z^2)^2} \cdot (0 + 2y + 0)$$

$$\frac{\partial^2 U}{\partial y \partial z} = \frac{-2z}{(x^2 + y^2 + z^2)^2} \cdot 2y$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{-4yz}{(x^2+y^2+z^2)^2}$$

b. side multiply with x

$$x \cdot \frac{\partial^2 u}{\partial y \partial z} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- ①}$$

$$u = \log(x^2+y^2+z^2)$$

$$\frac{\partial^2 u}{\partial z \partial x}$$

P.D wrto x

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} [\log(x^2+y^2+z^2)]$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2} \cdot \frac{\partial}{\partial x} (x^2+y^2+z^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2} \cdot [2x+0+0]$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2+z^2}$$

Again PD wrto z

$$\frac{\partial}{\partial z} \left[\frac{\partial u}{\partial x} \right] = \frac{\partial}{\partial z} \left[\frac{2x}{x^2+y^2+z^2} \right]$$

$$\frac{\partial^2 u}{\partial z \partial x} = 2x \cdot \frac{\partial}{\partial z} \left[\frac{1}{x^2+y^2+z^2} \right]$$

$$\frac{\partial^2 u}{\partial z \partial x} = 2x \cdot \frac{-1}{(x^2+y^2+z^2)^2} \cdot \frac{\partial}{\partial z} (x^2+y^2+z^2)$$

$$\frac{\partial^2 u}{\partial z \partial x} = \frac{-2x}{(x^2+y^2+z^2)^2} \cdot [0+0+2z]$$

$$\frac{\partial^2 u}{\partial z \partial x} = \frac{-4xz}{(x^2+y^2+z^2)^2}$$

multiply b.s with y

$$y \cdot \frac{\partial^2 u}{\partial z \partial x} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- (2)}$$

$$u = \log(x^2+y^2+z^2)$$

$$\frac{\partial^2 u}{\partial x \partial y}$$

P.D w.r.to y

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (\log(x^2+y^2+z^2))$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2+z^2} \cdot \frac{\partial}{\partial y} (x^2+y^2+z^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2+z^2} (0+2y+0)$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2+z^2}$$

Again P.D w.r.to x

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = 2xy \frac{\partial}{\partial x} \left(\frac{1}{x^2+y^2+z^2} \right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2y \frac{(-1)}{(x^2+y^2+z^2)^2} \cdot \frac{\partial}{\partial x} (x^2+y^2+z^2)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-2xy}{(x^2+y^2+z^2)^2} \cdot [2x+0+0]$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-4xy}{(x^2+y^2+z^2)^2}$$

multiply bs with z

$$z \cdot \frac{\partial^2 u}{\partial x \partial y} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- (3)}$$

from eq (1) (2) (3)

$$x \cdot \frac{\partial^2 u}{\partial y \partial z} = y \cdot \frac{\partial^2 u}{\partial z \partial x} = z \cdot \frac{\partial^2 u}{\partial x \partial y}$$

Hence proved.

② If $z = \tan^{-1} \left(\frac{y}{x} \right)$ verify that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Given,

$$z = \tan^{-1} \left(\frac{y}{x} \right)$$

P.D wrto x

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{\partial}{\partial x} \left(\frac{y}{x} \right)$$

$$\frac{\partial z}{\partial x} = \frac{1}{\frac{x^2 + y^2}{x^2}} \cdot y \cdot \frac{\partial}{\partial x} \left(\frac{1}{x} \right)$$

$$\frac{\partial z}{\partial x} = \frac{x^2}{x^2 + y^2} \cdot y \cdot \frac{-1}{x^2}$$

$$\frac{\partial z}{\partial x} = \frac{-y}{x^2 + y^2}$$

P.D wrto x

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{-y}{x^2 + y^2} \right)$$

$$\frac{\partial^2 z}{\partial x^2} = -y \left[\frac{-1}{(x^2 + y^2)^2} \cdot \frac{\partial}{\partial x} (x^2 + y^2) \right]$$

$$\frac{\partial^2 z}{\partial x^2} = -y \left[\frac{-1}{(x^2 + y^2)^2} \cdot [2x + 0] \right]$$

$$\frac{\partial^2 z}{\partial x^2} = -y \left[\frac{-2x}{(x^2 + y^2)^2} \right]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2} \quad \text{--- (P)}$$

$$z = \tan^{-1}\left(\frac{y}{x}\right)$$

p.d wrto y

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x}(1)$$

$$\frac{\partial z}{\partial y} = \frac{1}{\frac{x^2 + y^2}{x^2}} \cdot \frac{1}{x}$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x}$$

$$\frac{\partial z}{\partial y} = \frac{x}{x^2 + y^2}$$

Again wrto y

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = x \cdot \frac{\partial}{\partial y} \left(\frac{1}{x^2 + y^2} \right)$$

$$\frac{\partial^2 z}{\partial y^2} = x \cdot \frac{-1}{(x^2 + y^2)^2} \cdot \frac{\partial}{\partial y} (x^2 + y^2)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-x}{(x^2 + y^2)^2} \cdot [0 + 2y]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \quad (2)$$

Adding eq (1) & (2)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 //$$

Hence proved

* If $Z = \tan(y+ax) + (y-ax)^{3/2}$

find the value of $\frac{\partial^2 z}{\partial x^2} + a^2 \frac{\partial^2 z}{\partial y^2} = 0$

Given,

$$Z = \tan(y+ax) + (y-ax)^{3/2}$$

p.d w.r.to x

$$\frac{\partial z}{\partial x} = \sec^2(y+ax) \cdot \frac{\partial}{\partial x}(y+ax) + \frac{3}{2} (y-ax)^{3/2-1} \cdot \frac{\partial}{\partial x}(y-ax)$$

$$\frac{\partial z}{\partial x} = \sec^2(y+ax) (0+a(1)) + \frac{3}{2} (y-ax)^{1/2} (0-a(1))$$

$$\frac{\partial z}{\partial x} = \sec^2(y+ax) a + \frac{3}{2} (y-ax)^{1/2} (-a)$$

$$\frac{\partial z}{\partial x} = \sec^2(y+ax) a - \frac{3a}{2} (y-ax)^{1/2}$$

$$\frac{\partial z}{\partial x} = \sec^2(y+ax)a - \frac{3a}{2}(y-ax)^{\frac{1}{2}}$$

Again P.D w.r.to x

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = a \cdot 2 \sec(y+ax) \cdot \frac{\partial}{\partial x}(y+ax) - \frac{3a}{2} \left(\frac{1}{2} (y-ax)^{-\frac{1}{2}} \right)$$

$$\frac{\partial}{\partial x} (y-ax)$$

$$\frac{\partial^2 z}{\partial x^2} = 2a \sec(y+ax) (0+a(1)) - \frac{3a}{2} \left(\frac{1}{2} (y-ax)^{-\frac{1}{2}} (0-a(1)) \right)$$

$$\frac{\partial^2 z}{\partial x^2} = 2a^2 \sec(y+ax) + \frac{3a^2}{4} (y-ax)^{-\frac{1}{2}}$$

$$\frac{\partial^2 z}{\partial x^2} = a^2 \left[2 \sec(y+ax) + \frac{3}{4} (y-ax)^{-\frac{1}{2}} \right] \quad \text{--- (1)}$$

$$z = \tan(y+ax) + (y-ax)^{\frac{3}{2}}$$

P.D w.r.to y

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) \cdot \frac{\partial}{\partial y}(y+ax) + \frac{3}{2} (y-ax)^{\frac{3}{2}-1} \cdot \frac{\partial}{\partial y}(y-ax)$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) (1+a(0)) + \frac{3}{2} (y-ax)^{\frac{1}{2}} \cdot (1-0)$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) (1) + \frac{3}{2} (y-ax)^{\frac{1}{2}}$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{-1/2}$$

Again PD wrto y

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = 2 \sec(y+ax) \cdot \frac{\partial}{\partial y} (y+ax) + \frac{3}{2} \left[\frac{1}{2} (y-ax)^{-1/2} \cdot \frac{\partial}{\partial y} (y-ax) \right]$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec(y+ax) (1+0) + \frac{3}{2} \left[\frac{1}{2} (y-ax)^{-1/2} (1-0) \right]$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec(y+ax) + \frac{3}{4} (y-ax)^{-1/2}$$

multiple a^2 on b.s

$$a^2 \cdot \frac{\partial^2 z}{\partial y^2} = a^2 \left[2 \sec(y+ax) + \frac{3}{4} (y-ax)^{-1/2} \right] \quad \text{--- (2)}$$

sub eq ① & ②

$$\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = a^2 \left[2 \sec(y+ax) + \frac{3}{4} (y-ax)^{-1/2} \right] - a^2 \left[2 \sec(y+ax) + \frac{3}{4} (y-ax)^{-1/2} \right]$$

$$\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0 //$$

Hence proved.

④ If $u = \log(\sec x + \sec y + \sec z)$. Then S.T

$$\sum \cot x \frac{\partial u}{\partial x} = 1$$

Given $u = \log(\sec x + \sec y + \sec z)$

PD wrto x

$$\frac{\partial u}{\partial x} = \frac{1}{\sec x + \sec y + \sec z} \cdot \frac{\partial}{\partial x} (\sec x + \sec y + \sec z)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sec x + \sec y + \sec z} \cdot (\sec x \tan x + 0 + 0)$$

$$\frac{\partial u}{\partial x} = \frac{\sec x \tan x}{\sec x + \sec y + \sec z}$$

M.B sides by $\cot x$

$$\cot x \cdot \frac{\partial u}{\partial x} = \frac{\cot x \cdot \sec x \cdot \tan x}{\sec x + \sec y + \sec z}$$

$$= \frac{1}{\cancel{\tan x}} \cdot \frac{\sec x \cancel{\tan x}}{\sec x + \sec y + \sec x}$$

$$\cot x \cdot \frac{\partial u}{\partial x} = \frac{\sec x}{\sec x + \sec y + \sec x} \quad \text{--- (1)}$$

Similarly, $\cot y \cdot \frac{\partial u}{\partial y} = \frac{\sec y}{\sec x + \sec y + \sec x} \quad \text{--- (2)}$

$$\cot z \cdot \frac{\partial v}{\partial z} = \frac{\sec z}{\sec x + \sec y + \sec z} \quad (3)$$

Adding ① ② ③

$$\cot x \cdot \frac{\partial v}{\partial x} + \cot y \cdot \frac{\partial v}{\partial y} + \cot z \cdot \frac{\partial v}{\partial z} = \frac{\sec x + \sec y + \sec z}{\sec x + \sec y + \sec z}$$

$$\Sigma \cot x \cdot \frac{\partial v}{\partial x} = 1 //$$

Hence proved.

2020 (4m)

① If $f(x, y) = \frac{2y}{y + \cos x}$. Then evaluate $\frac{\partial^2 f}{\partial x \partial y}$

Given, $f = \frac{2y}{y + \cos x}$

P.D wrto y

$$\frac{\partial f}{\partial y} = \frac{y + \cos x \cdot \frac{\partial}{\partial y} (2y) - 2y \cdot \frac{\partial}{\partial y} (y + \cos x)}{(y + \cos x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{y + \cos x \cdot [2(1)] - 2y [1 + 0]}{(y + \cos x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(y + \cos x) 2 - 2y}{(y + \cos x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2(y + \cos x - y)}{(y + \cos x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2 \cos x}{(y + \cos x)^2}$$

Again P.D w.r.to x

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{(y + \cos x)^2 \cdot \frac{\partial}{\partial x} (2 \cos x) - 2 \cos x \cdot \frac{\partial}{\partial x} (y + \cos x)^2}{((y + \cos x)^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(y + \cos x)^2 \cdot 2(-\sin x) - 2 \cos x \cdot 2(y + \cos x) \cdot \frac{\partial}{\partial x} (y + \cos x)}{(y + \cos x)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2(y + \cos x)^2 (-\sin x) - 2 \cos x \cdot 2(y + \cos x) \cdot [0 + (-\sin x)]}{(y + \cos x)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2(y + \cos x)^2 (-\sin x) - 2 \cos x \cdot 2(y + \cos x) (-\sin x)}{(y + \cos x)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(y + \cos x) [2(y + \cos x) (-\sin x) + 4 \cos x \sin x]}{(y + \cos x)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2(y + \cos x) (-\sin x) + 4(\cos x \sin x)}{(y + \cos x)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-2y \sin x - 2 \cos x \sin x + 4 \cos x \sin x}{(y + \cos x)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-2y \sin x + 2 \cos x \sin x}{(y + \cos x)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-2 \sin x (y - \cos x)}{(y + \cos x)^3} //$$

② If $f(x, y) = \log(x^2 + xy + y^2)$

evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Given, $f = \log(x^2 + xy + y^2)$

P.D w.r.to x

$$\frac{\partial f}{\partial x} = \frac{1}{x^2 + xy + y^2} \cdot \frac{\partial}{\partial x}(x^2 + xy + y^2)$$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2 + xy + y^2} [2x + y(1) + 0]$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{2x + y}{x^2 + xy + y^2}}$$

$$f = \log(x^2 + xy + y^2)$$

P.D w.r.to y

$$\frac{\partial f}{\partial y} = \frac{1}{x^2 + xy + y^2} \cdot \frac{\partial}{\partial y}(x^2 + xy + y^2)$$

$$= \frac{1}{x^2 + xy + y^2} \cdot [0 + x(1) + 2y]$$

$$\boxed{\frac{\partial}{\partial y} = \frac{x+2y}{x^2+xy+y^2}}$$

Q If $U = e^x [x \cos y - y \sin y]$ Then S-T

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

Given

$$U = e^x [x \cos y - y \sin y]$$

PD wrto x

$$\frac{\partial U}{\partial x} = e^x \frac{\partial}{\partial x} (x \cos y - y \sin y) + (x \cos y - y \sin y) \frac{\partial}{\partial x} (e^x)$$

$$\frac{\partial U}{\partial x} = e^x [\cos y (1) - 0] + (x \cos y - y \sin y) e^x$$

$$\frac{\partial U}{\partial x} = e^x \cos y + (x \cos y - y \sin y) e^x$$

$$\frac{\partial U}{\partial x} = e^x [\cos y + x \cos y - y \sin y]$$

Again PD wrto x

$$\frac{\partial^2 U}{\partial x^2} = e^x \frac{\partial}{\partial x} (\cos y + x \cos y - y \sin y) + (\cos y + x \cos y - y \sin y) \frac{\partial}{\partial x} (e^x)$$

$$\frac{\partial^2 U}{\partial x^2} = e^x [0 + \cos y (1) - 0] + (\cos y + x \cos y - y \sin y) e^x$$

$$\frac{\partial^2 u}{\partial x^2} = e^x (\cos y) + (\cos y + x \cos y - y \sin y) e^x$$

$$\frac{\partial^2 u}{\partial x^2} = e^x [\cos y + \cos y + x \cos y - y \sin y]$$

$$\frac{\partial^2 u}{\partial x^2} = e^x [2 \cos y + x \cos y - y \sin y] \text{ --- ①}$$

$$u = e^x [x \cos y - y \sin y]$$

P.D. wrto y

$$\frac{\partial u}{\partial y} = e^x \cdot \frac{\partial}{\partial y} [x \cos y - y \sin y]$$

$$\frac{\partial u}{\partial y} = e^x \left[x(-\sin y) - \left[y \cdot \frac{\partial}{\partial y} (\sin y) + \sin y \cdot \frac{\partial}{\partial y} (y) \right] \right]$$

$$\frac{\partial u}{\partial y} = e^x \left[x(-\sin y) - [y(\cos y) + \sin y \cdot (1)] \right]$$

$$\frac{\partial u}{\partial y} = e^x [x(-\sin y) - y \cos y - \sin y]$$

$$\frac{\partial u}{\partial y} = e^x [-x \sin y - y \cos y - \sin y]$$

Again P.D wrto y

$$\frac{\partial^2 u}{\partial y^2} = e^x \frac{\partial}{\partial y} [-x \sin y - y \cos y - \sin y]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x \left[-x \left(\frac{\partial}{\partial y} (\sin y) \right) - \left[y \cdot \frac{\partial}{\partial y} (\cos y) + \cos y \frac{\partial}{\partial y} (y) \right] - \frac{\partial}{\partial y} (\sin y) \right]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x \left[-x (\cos y) - [y (-\sin y) + \cos y (1)] - \cos y \right]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x \left[-x \cos y + y \sin y - \cos y - \cos y \right]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x \left[-x \cos y + y \sin y + 2 \cos y \right] \quad (2)$$

Adding eq (1) & (2)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x [2 \cos y + x \cos y - y \sin y] + e^x [-x \cos y + y \sin y - 2 \cos y]$$

$$= e^x \cancel{2 \cos y} + e^x \cancel{x \cos y} - e^x y \sin y - e^x \cancel{x \cos y} + e^x y \sin y - e^x \cancel{2 \cos y}$$

$$= 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Hence proved //

H.W
① find the first order partial derivatives of

(i) $\tan^{-1}(x+y)$

(ii) $a^x \sin by$

(iii) $\log(x^2+y^2)$

(i) $\tan^{-1}(x+y)$

$$f = \tan^{-1}(x+y)$$

p.d w.r.to x

$$\frac{\partial f}{\partial x} = \frac{1}{1+(x+y)^2} \cdot \frac{\partial}{\partial x}(x+y)$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+(x+y)^2} \cdot (1+0)$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{1}{1+(x+y)^2}}$$

$$f = \tan^{-1}(x+y)$$

p.d w.r.to y

$$\frac{\partial f}{\partial y} = \frac{1}{1+(x+y)^2} \cdot \frac{\partial}{\partial y}(x+y)$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(x+y)^2} \cdot (0+1)$$

$$\boxed{\frac{\partial f}{\partial y} = \frac{1}{1+(x+y)^2}}$$

$$(ii) e^{ax} \sin by$$

$$f = e^{ax} \sin by$$

p.d w.r.to x

$$\frac{\partial f}{\partial x} = \sin by \cdot \frac{\partial}{\partial x} (e^{ax})$$

$$\frac{\partial f}{\partial x} = \sin by \cdot e^{ax} \cdot \frac{\partial}{\partial x} (ax)$$

$$\frac{\partial f}{\partial x} = \sin by \cdot e^{ax} \cdot a(1)$$

$$\boxed{\frac{\partial f}{\partial x} = a e^{ax} \sin by}$$

$$f = e^{ax} \sin by$$

p.d w.r.to y

$$\frac{\partial f}{\partial y} = e^{ax} \cdot \frac{\partial}{\partial y} (\sin by)$$

$$\frac{\partial f}{\partial y} = e^{ax} (\cos by) \cdot \frac{\partial}{\partial y} (by)$$

$$\frac{\partial f}{\partial y} = e^{ax} (\cos by) b(1)$$

$$\boxed{\frac{\partial f}{\partial y} = b e^{ax} \cos by}$$

$$(iii) \log(x^2 + y^2)$$

$$f = \log(x^2 + y^2)$$

p.d w.r.to x

$$\frac{\partial f}{\partial x} = \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial x} (x^2 + y^2)$$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2 + y^2} (2x + 0)$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}}$$

$$f = \log(x^2 + y^2)$$

p.d w.r.to y

$$\frac{\partial f}{\partial y} = \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial y} (x^2 + y^2)$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2 + y^2} (0 + 2y)$$

$$\boxed{\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}}$$

② find the second order partial derivative.

(i) e^{x-y} (ii) e^{x^y} (iii) $\tan[\tan^{-1}x + \tan^{-1}y]$

(i) e^{x-y}

$$f = e^{x-y}$$

P.D wrto x

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{x-y})$$

$$\frac{\partial f}{\partial x} = e^{x-y} \cdot \frac{\partial}{\partial x}(x-y)$$

$$\frac{\partial f}{\partial x} = e^{x-y} [1-0]$$

$$\frac{\partial f}{\partial x} = e^{x-y}$$

Again P.D wrto x

$$\frac{\partial^2 f}{\partial x^2} = e^{x-y} \cdot \frac{\partial}{\partial x}(x-y)$$

$$\frac{\partial^2 f}{\partial x^2} = e^{x-y} (1-0)$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = e^{x-y}}$$

$$f = e^{x-y}$$

P.D wrto y

$$\frac{\partial f}{\partial y} = e^{x-y} \cdot \frac{\partial}{\partial y}(x-y)$$

$$\frac{\partial f}{\partial y} = e^{x-y} (0-1)$$

$$\frac{\partial f}{\partial y} = -e^{x-y}$$

Again P.D wrto y

$$\frac{\partial^2 f}{\partial y^2} = -e^{x-y} \cdot \frac{\partial}{\partial y}(x-y)$$

$$\frac{\partial^2 f}{\partial y^2} = -e^{x-y} (0-1)$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = e^{x-y}}$$

$$f = e^{x-y}$$

p.d w.r.to x

$$\frac{\partial f}{\partial x} = e^{x-y} \cdot \frac{\partial}{\partial x}(x-y)$$

$$\frac{\partial f}{\partial x} = e^{x-y}(1-0)$$

$$\frac{\partial f}{\partial x} = e^{x-y}$$

p.d w.r.to y

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(e^{x-y})$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^{x-y} \cdot \frac{\partial}{\partial y}(x-y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^{x-y}(0-1)$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = -e^{x-y}}$$

$$f = e^{x-y}$$

p.d w.r.to y

$$\frac{\partial f}{\partial y} = e^{x-y} \cdot \frac{\partial}{\partial y}(x-y)$$

$$\frac{\partial f}{\partial y} = e^{x-y}(0-1)$$

$$\frac{\partial f}{\partial y} = -e^{x-y}$$

p.d w.r.to x

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -e^{x-y} \cdot \frac{\partial}{\partial x}(x-y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -e^{x-y}(1-0)$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = -e^{x-y}}$$

$$(iii) \tan^{-1} [\tan^{-1} x + \tan^{-1} y]$$

$$f = \tan [\tan^{-1} x + \tan^{-1} y]$$

$$f = \tan \left[\tan^{-1} \left[\frac{x+y}{1-xy} \right] \right]$$

$$f = \frac{x+y}{1-xy}$$

p.d wrto x

$$\frac{\partial f}{\partial x} = \frac{(1-xy) \frac{\partial}{\partial x}(x+y) - (x+y) \frac{\partial}{\partial x}(1-xy)}{(1-xy)^2}$$

$$\frac{\partial f}{\partial x} = \frac{(1-xy)(1+0) - (x+y)(0-xy(1))}{(1-xy)^2}$$

$$\frac{\partial f}{\partial x} = \frac{(1-xy) - (x+y)(-y)}{(1-xy)^2}$$

$$\frac{\partial f}{\partial x} = \frac{1-xy+xy+y^2}{(1-xy)^2}$$

$$\frac{\partial f}{\partial x} = \frac{1+y^2}{(1-xy)^2} \quad \text{--- (1)}$$

Again p.d wrto x

$$\frac{\partial^2 f}{\partial x^2} = 1+y^2 \cdot \frac{\partial}{\partial x} \left(\frac{1}{(1-xy)^2} \right)$$

$$\frac{\partial^2 f}{\partial x^2} = 1+y^2 \frac{\partial}{\partial x} ((1-xy)^{-2})$$

$$\frac{\partial^2 f}{\partial x^2} = 1+y^2 [-2(1-xy)^{-3-1}] \cdot \frac{\partial}{\partial x} (1-xy)$$

$$\frac{\partial^2 f}{\partial x^2} = 1+y^2 [-2(1-xy)^{-3}] [0-y(1)]$$

$$\frac{\partial^2 f}{\partial x^2} = 1+y^2 (-2) \frac{1}{(1-xy)^3} [-y]$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{2y(1+y^2)}{(1-xy)^3}}$$

$$f = \frac{x+y}{1-xy}$$

p.d w.r.to y

$$\frac{\partial f}{\partial y} = \frac{(1-xy) \frac{\partial}{\partial y} (x+y) - (x+y) \frac{\partial}{\partial y} (1-xy)}{(1-xy)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(1-xy)(0+1) - (x+y)(0-x(1))}{(1-xy)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(1-xy) - (x+y)(-x)}{(1-xy)^2}$$

$$\frac{\partial f}{\partial y} = \frac{1-xy+x^2+xy}{(1-xy)^2}$$

$$\frac{\partial f}{\partial y} = \frac{1+x^2}{(1-xy)^2} \quad \text{--- (2)}$$

Again P.D w.r.to y

$$\frac{\partial^2 f}{\partial y^2} = (1+x^2) \frac{\partial}{\partial y} \left(\frac{1}{(1-xy)^2} \right)$$

$$\frac{\partial^2 f}{\partial y^2} = 1+x^2 \cdot \frac{\partial}{\partial y} (1-xy)^{-2}$$

$$\frac{\partial^2 f}{\partial y^2} = 1+x^2 \cdot [-2(1-xy)^{-2-1}] \cdot \frac{\partial}{\partial y} (1-xy)$$

$$\frac{\partial^2 f}{\partial y^2} = 1+x^2 [-2(1-xy)^{-3}] \cdot (0-x(1))$$

$$\frac{\partial^2 f}{\partial y^2} = 1+x^2 (-2) \frac{1}{(1-xy)^3} (-x)$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = \frac{2x(1+x^2)}{(1-xy)^3}}$$

from eq (1) $\frac{\partial f}{\partial x} = \frac{1+y^2}{(1-xy)^2}$

P.D w.r.to y

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{1+y^2}{(1-xy)^2} \right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(1-xy)^2 \frac{\partial}{\partial y} (1+y^2) - (1+y^2) \frac{\partial}{\partial y} (1-xy)^2}{((1-xy)^2)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(1-xy)^2 (0+2y) - (1+y^2) 2(1-xy) \cdot \frac{\partial}{\partial y} (1-xy)}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(1-xy)^2 2y - (1+y^2) 2(1-xy) (0-x(1))}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(1-xy)^2 2y - (1+y^2) 2(1-xy) (-x)}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(1-xy)^2 2y + (1+y^2) 2(1-xy) x}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(1-xy) [(1-xy) 2y + (1+y^2) 2x]}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(1-xy) 2y + (1+y^2) 2x}{(1-xy)^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{2y - 2xy^2 + 2x + 2xy^2}{(1-xy)^3}$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{2(x+y)}{(1-xy)^3}}$$

from eq (2) $\frac{\partial f}{\partial y} = \frac{1+x^2}{(1-xy)^2}$

P.D wrto x

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{1+x^2}{(1-xy)^2} \right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(1-xy)^2 \frac{\partial}{\partial x} (1+x^2) - (1+x^2) \frac{\partial}{\partial x} (1-xy)^2}{((1-xy)^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(1-xy)^2 (0+2x) - (1+x^2) 2(1-xy) \frac{\partial}{\partial x} (1-xy)}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(1-xy)^2 2x - (1+x^2) 2(1-xy) (0-y(1))}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(1-xy)^2 2x - (1+x^2) 2(1-xy) (-y)}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(1-xy) [(1-xy) 2x - (1+x^2) 2(-y)]}{(1-xy)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2x - 2xy + 2y + 2xy}{(1-xy)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2x+2y}{(1-xy)^3}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{2(x+y)}{(1-xy)^3}}$$

③ Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, when u is

(i) $\sin^{-1} \frac{x}{y}$

(ii) $\frac{xy}{\sqrt{1+x^2+y^2}}$

(iii) $\log(y \sin x + x \sin y)$

(i) $\sin^{-1} \left(\frac{x}{y} \right)$

$u = \sin^{-1} \left(\frac{x}{y} \right)$

p.d w.r.to y

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left[\sin^{-1} \left(\frac{x}{y} \right) \right]$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \left(\frac{x}{y} \right)^2}} \cdot \frac{\partial}{\partial y} \left(\frac{x}{y} \right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot x \cdot \frac{\partial}{\partial y} \left(\frac{1}{y} \right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{\frac{y^2 - x^2}{y^2}}} \cdot x \cdot \left(\frac{-1}{y^2} \right)$$

$$\frac{\partial v}{\partial y} = \frac{1}{\frac{\sqrt{y^2-x^2}}{\sqrt{y^2}}} \cdot x \left(\frac{-1}{y^2} \right)$$

$$\frac{\partial v}{\partial y} = \frac{y}{\sqrt{y^2-x^2}} \cdot x \frac{-1}{y^2}$$

$$\frac{\partial v}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}}$$

Again pop wto x

$$\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = -\frac{1}{y} \cdot \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{y^2-x^2}} \right)$$

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{1}{y} \cdot \left[\frac{\sqrt{y^2-x^2} \cdot \frac{\partial}{\partial x}(x) - (x) \frac{\partial}{\partial x}(\sqrt{y^2-x^2})}{(\sqrt{y^2-x^2})^2} \right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{1}{y} \left[\frac{\sqrt{y^2-x^2} (1) - (x) \cdot \frac{1}{2\sqrt{y^2-x^2}} \cdot \frac{\partial}{\partial x}(y^2-x^2)}{y^2-x^2} \right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{1}{y} \left[\frac{\sqrt{y^2-x^2} - (x) \frac{1}{2\sqrt{y^2-x^2}} \cdot (0-2x)}{y^2-x^2} \right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{1}{y} \left[\frac{\sqrt{y^2-x^2} + \frac{2x^2}{2\sqrt{y^2-x^2}}}{y^2-x^2} \right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{-1}{y} \left[\frac{\sqrt{y^2 - x^2} + \frac{x}{\sqrt{y^2 - x^2}}}{y^2 - x^2} \right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{-1}{y} \left[\frac{(\sqrt{y^2 - x^2})^x + x^2}{\sqrt{y^2 - x^2}} \cdot \frac{1}{y^2 - x^2} \right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{-1}{y} \left[\frac{y^x - x^2 + x^2}{\sqrt{y^2 - x^2} y^2 - x^2} \right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{-y}{(y^2 - x^2)^{3/2} (y^2 - x^2)}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{-y}{(y^2 - x^2)^{5/2}}$$

$$\boxed{\frac{\partial^2 v}{\partial x \partial y} = \frac{-y}{(y^2 - x^2)^{3/2}}} \quad \text{--- (1)}$$

$$v = \sin^{-1} \left(\frac{x}{y} \right)$$

P.D w.r.to y^x

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x} \left(\frac{x}{y} \right)$$

$$\frac{\partial}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot y \cdot \frac{\partial}{\partial x} (x)$$

$$U = \sin^{-1}\left(\frac{x}{y}\right)$$

P.D. wrt to x

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{1}{y} \cdot \frac{\partial}{\partial x}(x)$$

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{\frac{y^2 - x^2}{y^2}}} \cdot \frac{1}{y} (1)$$

$$\frac{\partial U}{\partial x} = \frac{1}{\frac{\sqrt{y^2 - x^2}}{\sqrt{y^2}}} \cdot \frac{1}{y}$$

$$\frac{\partial U}{\partial x} = \frac{y}{\sqrt{y^2 - x^2}} \cdot \frac{1}{y}$$

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}}$$

P.D. wrt to y

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{1}{(y^2 - x^2)^{1/2}} \right)$$

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y} \left((y^2 - x^2)^{-1/2} \right)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-1}{2} (y^2 - x^2)^{-1/2} \cdot \frac{\partial}{\partial y} (y^2 - x^2)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-1}{2} (y^2 - x^2)^{-3/2} (2y - 0)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-xy}{(y^2 - x^2)^{3/2}}$$

$$\boxed{\frac{\partial^2 u}{\partial y \partial x} = \frac{-y}{(y^2 - x^2)^{3/2}}} \quad \text{--- (2)}$$

equating eq (1) & (2)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Hence proved. //

$$(11) \frac{xy}{\sqrt{1+x^2+y^2}}$$

$$u = \frac{xy}{\sqrt{1+x^2+y^2}}$$

PD wrto y

$$\frac{\partial u}{\partial y} = \frac{\sqrt{1+x^2+y^2} \cdot \frac{\partial}{\partial y} (xy) - xy \cdot \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{1+x^2+y^2}} \right)}{(\sqrt{1+x^2+y^2})^2}$$

$$\frac{\partial v}{\partial y} = \frac{\sqrt{1+x^2+y^2} \cdot x(1) - xy \cdot \left[\frac{1}{2\sqrt{1+x^2+y^2}} \cdot \frac{\partial}{\partial y}(1+x^2+y^2) \right]}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{\sqrt{1+x^2+y^2} \cdot x - xy \left[\frac{1}{2\sqrt{1+x^2+y^2}} \cdot [0+0+2y] \right]}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{\sqrt{1+x^2+y^2} \cdot x - xy^2 \left[\frac{1}{\sqrt{1+x^2+y^2}} \right]}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{\sqrt{1+x^2+y^2} \cdot x - \frac{xy^2}{\sqrt{1+x^2+y^2}}}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{\frac{(\sqrt{1+x^2+y^2})^2 x - xy^2}{\sqrt{1+x^2+y^2}}}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{(1+x^2+y^2)x - xy^2}{(1+x^2+y^2)^{3/2}} \cdot \frac{1}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{x + x^3 + xy^2 - xy^2}{(1+x^2+y^2)^{5/2}}$$

$$\frac{\partial v}{\partial y} = \frac{x + x^3}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial v}{\partial y} = \frac{x+x^3}{(1+x^2+y^2)^{3/2}}$$

Again w.r.to x

$$\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = \frac{(1+x^2+y^2)^{3/2} \cdot \frac{\partial}{\partial x} (x+x^3) - (x+x^3) \frac{\partial}{\partial x} (1+x^2+y^2)^{3/2}}{\left[(1+x^2+y^2)^{3/2} \right]^2}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{(1+x^2+y^2)^{3/2} [1+3x^2] - (x+x^3) \left[\frac{3}{2} (1+x^2+y^2)^{1/2} \right] \frac{\partial}{\partial x} (1+x^2+y^2)}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{(1+x^2+y^2)^{3/2} (1+3x^2) - (x+x^3) \left[\frac{3}{2} (1+x^2+y^2)^{1/2} \right] [0+2x+0]}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{(1+x^2+y^2)^{3/2} (1+3x^2) - (x+x^3) [3(1+x^2+y^2)^{1/2}] [x]}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{(1+x^2+y^2)^{1/2} [(1+x^2+y^2)(1+3x^2) - (x+x^3) 3x]}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{(1+x^2+y^2)^{1/2} [1+3x^2+x^2+3x^4+y^2+3x^2y^2-3x^2-3x^4]}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{(1+x^2+y^2)^{1/2} [1+x^2+y^2+3x^2y^2]}{(1+x^2+y^2)^3} \quad \text{--- ①}$$

$$v = \frac{xy}{\sqrt{1+x^2+y^2}}$$

PD wrto x

$$\frac{\partial v}{\partial x} = \frac{\sqrt{1+x^2+y^2} \cdot \frac{\partial}{\partial x}(xy) - xy \cdot \frac{\partial}{\partial x}(\sqrt{1+x^2+y^2})}{(\sqrt{1+x^2+y^2})^2}$$

$$\frac{\partial v}{\partial x} = \frac{\sqrt{1+x^2+y^2} \cdot y(1) - xy \cdot \frac{1}{2\sqrt{1+x^2+y^2}} \cdot \frac{\partial}{\partial x}(1+x^2+y^2)}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{\sqrt{1+x^2+y^2} \cdot y - xy \cdot \frac{1}{2\sqrt{1+x^2+y^2}} \cdot (0+2x+0)}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{\sqrt{1+x^2+y^2} \cdot y - \frac{x^2 y}{\sqrt{1+x^2+y^2}}}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{(\sqrt{1+x^2+y^2})^2 \cdot y - x^2 y}{\sqrt{1+x^2+y^2}} \cdot \frac{1}{1+x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{y + \cancel{x^2 y} + y^3 - \cancel{x^2 y}}{\sqrt{1+x^2+y^2} \cdot 1+x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{y+y^3}{(1+x^2+y^2)^{3/2}}$$

Again PD wrto y

$$\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{(1+x^2+y^2)^{3/2} \frac{\partial}{\partial y} (y+y^3) - (y+y^3) \frac{\partial}{\partial y} (1+x^2+y^2)^{3/2}}{[1+x^2+y^2]^{3/2}}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(1+x^2+y^2)^{3/2} [1+3y^2] - (y+y^3) \left[\frac{3}{2} (1+x^2+y^2)^{1/2} \right]}{\frac{\partial}{\partial y} (1+x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(1+x^2+y^2)^{3/2} [1+3y^2] - [y+y^3] \frac{3}{2} (1+x^2+y^2)^{1/2} (0+0+xy)}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(1+x^2+y^2)^{1/2} [(1+x^2+y^2)(1+3y^2) - (y+y^3)3y]}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(1+x^2+y^2)^{1/2} [1+3y^2+x^2+3x^2y^2+y+3y^4-3y^4-3y^4]}{[1+x^2+y^2]^3}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(1+x^2+y^2)^{1/2} [1+x^2+y^2+3x^2y^2]}{[1+x^2+y^2]^3} \quad \text{--- (2)}$$

equating eq (1) & (2)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Hence proved //

$$(iii) \log(y \sin x + x \sin y)$$

$$u = \log(y \sin x + x \sin y)$$

P.D w.r.to y

$$\frac{\partial u}{\partial y} = \frac{1}{y \sin x + x \sin y} \cdot \frac{\partial}{\partial y} (y \sin x + x \sin y)$$

$$\frac{\partial u}{\partial y} = \frac{1}{y \sin x + x \sin y} \cdot \sin x (1) + x (\cos y)$$

$$\frac{\partial u}{\partial y} = \frac{\sin x + x \cos y}{y \sin x + x \sin y}$$

Again p.D w.r.to x

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{y \sin x + x \sin y \cdot \frac{\partial}{\partial x} (\sin x + x \cos y) - (\sin x + x \cos y) \frac{\partial}{\partial x} (y \sin x + x \sin y)}{(y \sin x + x \sin y)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{y \sin x + x \sin y [\cos x + \cos y (1)] - [\sin x + x \cos y] [y (\cos x) + \sin y (1)]}{[y \sin x + x \sin y]^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{y \sin x + x \sin y [\cos x + \cos y] - [\sin x + x \cos y] [y \cos x + \sin y]}{(y \sin x + x \sin y)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{y \sin x \cos y + x \sin y \cos x - \sin x \sin y - x y \cos x \cos y}{(y \sin x + x \sin y)^2} \quad \text{--- ①}$$

P.

$$u = \log(y \sin x + x \sin y)$$

P.D w.r.to x

$$\frac{\partial u}{\partial x} = \frac{1}{y \sin x + x \sin y} \cdot \frac{\partial}{\partial x} (y \sin x + x \sin y)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y \sin x + x \sin y} [y(\cos x) + \sin y(1)]$$

$$\frac{\partial u}{\partial x} = \frac{y \cos x + \sin y}{y \sin x + x \sin y}$$

P.D w.r.to y

$$\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{y \sin x + x \sin y \frac{\partial}{\partial y} (y \cos x + \sin y) - (y \cos x + \sin y) \frac{\partial}{\partial y} (y \sin x + x \sin y)}{(y \sin x + x \sin y)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{y \sin x + x \sin y [\cos x(1) + \cos y] - (y \cos x + \sin y) [\sin x(1) + x(\cos y)]}{(y \sin x + x \sin y)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{y \sin x + x \sin y (\cos x + \cos y) - (y \cos x + \sin y) (\sin x + x \cos y)}{(y \sin x + x \sin y)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{y \sin x \cancel{\cos x} + y \sin x \cos y + x \sin y \cos x + x \sin y \cancel{\cos y} - [y \cos x \cancel{\sin x} + x y \cos x \cos y + \sin x \cancel{\sin y} + x \sin y \cancel{\cos y}]}{(y \sin x + x \sin y)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{y \sin x \cos y + x \sin y \cos x - x y \cos x \cos y - \sin x \sin y}{(y \sin x + x \sin y)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y \sin x \cos y + x \sin y \cos x - \sin x \sin y - x y \cos x \cos y}{(y \sin x + x \sin y)^2} \quad \text{--- (2)}$$

Equating eq (1) & (2)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Hence proved //

Q. If $z = 3xy - y^3 + (y^2 - 2x)^{3/2}$ verify that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2$$

Given, $z = 3xy - y^3 + (y^2 - 2x)^{3/2}$

P.D. wrto x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [3xy - y^3 + (y^2 - 2x)^{3/2}]$$

$$\frac{\partial z}{\partial x} = 3y(1) - 0 + \frac{3}{2} (y^2 - 2x)^{\frac{3}{2}-1} \cdot \frac{\partial}{\partial x} (y^2 - 2x)$$

$$\frac{\partial z}{\partial x} = 3y + \frac{3}{2} (y^2 - 2x)^{\frac{1}{2}} \cdot (0 - 2(1))$$

$$\frac{\partial z}{\partial x} = 3y + \frac{3}{2} (y^2 - 2x)^{\frac{1}{2}} (-2)$$

$$\frac{\partial z}{\partial x} = 3y - 3(y^2 - 2x)^{\frac{1}{2}} \quad \text{--- (1)}$$

Again P.D w.r.to x

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = 0 - 3 \left[\frac{1}{2} (y^2 - 2x)^{\frac{1}{2}-1} \cdot \frac{\partial}{\partial x} (y^2 - 2x) \right]$$

$$\frac{\partial^2 z}{\partial x^2} = -3 \cdot \frac{1}{2} (y^2 - 2x)^{-\frac{1}{2}} (0 - 2(1))$$

$$\frac{\partial^2 z}{\partial x^2} = +3 \cdot (y^2 - 2x)^{-\frac{1}{2}}$$

$$\frac{\partial^2 z}{\partial x^2} = 3 (y^2 - 2x)^{-\frac{1}{2}} \text{ --- (2)}$$

$$I = 3xy - y^3 + (y^2 - 2x)^{\frac{3}{2}}$$

P.D w.r.to y

$$\frac{\partial z}{\partial y} = 3x(1) - 3y^2 + \frac{3}{2} (y^2 - 2x)^{\frac{3}{2}-1} \cdot \frac{\partial}{\partial y} (y^2 - 2x)$$

$$\frac{\partial z}{\partial y} = 3x - 3y^2 + \frac{3}{2} (y^2 - 2x)^{\frac{1}{2}} \cdot (2y - 0)$$

$$\frac{\partial z}{\partial y} = 3x - 3y^2 + 3(y^2 - 2x)^{\frac{1}{2}} y$$

$$\frac{\partial z}{\partial y} = 3x - 3y^2 + 3y (y^2 - 2x)^{\frac{1}{2}}$$

$$\frac{\partial z}{\partial y} = 3 \left[x - y^2 + y(y^2 - 2x)^{\frac{1}{2}} \right] \text{ --- (3)}$$

Again w.r.to y

$$\frac{\partial^2 z}{\partial y^2} = 3 \cdot \frac{\partial}{\partial y} \left[x - y^2 + y(y^2 - 2x)^{\frac{1}{2}} \right]$$

$$\frac{\partial^2 z}{\partial y^2} = 3 \left[0 - 2y + y \frac{\partial}{\partial y} (y^2 - 2x)^{1/2} + (y^2 - 2x)^{1/2} \frac{\partial}{\partial y} (y) \right]$$

$$\frac{\partial^2 z}{\partial y^2} = 3 \left[-2y + y \left[\frac{1}{2} (y^2 - 2x)^{1/2 - 1} \cdot \frac{\partial}{\partial y} (y^2 - 2x) + (y^2 - 2x)^{1/2} (1) \right] \right]$$

$$\frac{\partial^2 z}{\partial y^2} = 3 \left[-2y + y \left[\frac{1}{2} (y^2 - 2x)^{-1/2} (2y - 0) + (y^2 - 2x)^{1/2} \right] \right]$$

$$\frac{\partial^2 z}{\partial y^2} = 3 \left[-2y + y^2 (y^2 - 2x)^{-1/2} + (y^2 - 2x)^{1/2} \right]$$

$$\frac{\partial^2 z}{\partial y^2} = -6y + 3y^2 (y^2 - 2x)^{-1/2} + 3(y^2 - 2x)^{1/2} \quad \text{--- (4)}$$

Now,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \quad \text{from eq (3)}$$

$$= \frac{\partial}{\partial x} \left[3 \left[x - y^2 + y (y^2 - 2x)^{1/2} \right] \right]$$

$$= 3 \cdot \frac{\partial}{\partial x} \left(x - y^2 + y (y^2 - 2x)^{1/2} \right)$$

$$= 3 \left[1 - 0 + y \cdot \frac{1}{2} (y^2 - 2x)^{1/2 - 1} \cdot \frac{\partial}{\partial x} (y^2 - 2x) \right]$$

$$= 3 \left[1 + y \cdot \frac{1}{2} (y^2 - 2x)^{-1/2} (0 - 2(1)) \right]$$

$$= 3 - 3y (y^2 - 2x)^{-1/2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3 - 3y (y^2 - 2x)^{-1/2} \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) \quad \text{from eq (1)}$$

$$= \frac{\partial}{\partial y} \left[3y - 3(y^2 - 2x)^{1/2} \right]$$

$$= 3(1) - 3 \left[\frac{1}{2} (y^2 - 2x)^{-1/2} \cdot \frac{\partial}{\partial y} (y^2 - 2x) \right]$$

$$= 3 - \frac{3}{2} (y^2 - 2x)^{-1/2} (2y - 0)$$

$$\frac{\partial^2 z}{\partial y \partial x} = 3 - 3y (y^2 - 2x)^{-1/2} \quad \text{--- (6)}$$

equating eq (5), (6)

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

Now,

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2$$

from
eq (2) (4)

from
eq (5)

$$3(y^2 - 2x)^{-1/2} \cdot (-6y) + 3y^2 (y^2 - 2x)^{-1/2} + 3(y^2 - 2x)^{1/2} = [3 - 3y(y^2 - 2x)^{-1/2}]^2$$

$$[3(y^2 - 2x)^{-1/2}]^2 \cdot (-6y) + 3y^2 (y^2 - 2x)^{-1/2} + 3(y^2 - 2x)^{1/2} = [3 - 3y(y^2 - 2x)^{-1/2}]^2$$

$$-18y (y^2 - 2x)^{-1/2} + 9y^2 (y^2 - 2x)^{-1/2} \cdot (y^2 - 2x)^{-1/2} + 9(y^2 - 2x)^{-1/2} \cdot (y^2 - 2x)^{1/2} \\ = (3)^2 + (3y(y^2 - 2x)^{-1/2})^2 - 2(3)(3y(y^2 - 2x)^{-1/2})$$

$$+18y(y^2-2x)^{-\frac{1}{2}} + 9y^2(y^2-2x)^{-\frac{1}{2}-1} + 9(y^2-2x)^{-\frac{1}{2}+1}$$

$$= 9 + 9y^2(y^2-2x)^{-1} - 18y(y^2-2x)^{-\frac{1}{2}}$$

$$-18y(y^2-2x)^{-\frac{1}{2}} + 9y^2(y^2-2x)^{-1} + 9(y^2-2x)^0$$

$$= 9 + 9y^2(y^2-2x)^{-1} - 18y(y^2-2x)^{-\frac{1}{2}}$$

$$-18y(y^2-2x)^{-\frac{1}{2}} + 9y^2(y^2-2x)^{-1} + 9(1)$$

$$= 9 + 9y^2(y^2-2x)^{-1} - 18y(y^2-2x)^{-\frac{1}{2}}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2$$

Hence proved //

Q) If $u = \tan^{-1} \left[\frac{xy}{\sqrt{1+x^2+y^2}} \right]$ then show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{vu' - uv'}{v^2}$$

Given

$$u = \tan^{-1} \left[\frac{xy}{\sqrt{1+x^2+y^2}} \right]$$

P.D w.r.to y

$$\frac{\partial v}{\partial y} = \frac{1}{1 + \left[\frac{xy}{\sqrt{1+x^2+y^2}} \right]^2} \cdot \frac{\partial}{\partial y} \left[\frac{xy}{\sqrt{1+x^2+y^2}} \right]$$

$$\frac{\partial v}{\partial y} = \frac{1}{1 + \frac{(xy)^2}{(\sqrt{1+x^2+y^2})^2}} \cdot \frac{\sqrt{1+x^2+y^2} \cdot \frac{d}{dy}(xy) - xy \frac{\partial}{\partial y}(\sqrt{1+x^2+y^2})}{(\sqrt{1+x^2+y^2})^2}$$

$$= \frac{1}{\frac{1+x^2+y^2+x^2y^2}{1+x^2+y^2}} \cdot \frac{\sqrt{1+x^2+y^2} \cdot x(1) - xy \cdot \frac{1}{2\sqrt{1+x^2+y^2}} \cdot \frac{\partial}{\partial y}(1+x^2+y^2)}{(1+x^2+y^2)}$$

$$= \frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2} \cdot \frac{\sqrt{1+x^2+y^2} \cdot x - xy \cdot \frac{1}{2\sqrt{1+x^2+y^2}} (0+0+2y)}{1+x^2+y^2}$$

$$= \frac{1}{1+x^2+y^2+x^2y^2} \cdot \left[\sqrt{1+x^2+y^2} \cdot x - xy^2 \cdot \frac{1}{\sqrt{1+x^2+y^2}} \right]$$

$$= \frac{1}{(1+x^2+y^2)(1+x^2)} \left[\frac{(\sqrt{1+x^2+y^2})^2 \cdot x - xy^2}{\sqrt{1+x^2+y^2}} \right]$$

$$= \frac{1}{(1+x^2)(1+y^2)} \left[\frac{(1+x^2+y^2)x - xy^2}{\sqrt{1+x^2+y^2}} \right]$$

$$= \frac{1}{(1+x^2)(1+y^2)} \left[\frac{x+x^3+xy^2-xy^3}{\sqrt{1+x^2+y^2}} \right]$$

$$= \frac{x+x^3}{(1+x^2)(1+y^2)\sqrt{1+x^2+y^2}}$$

$$= \frac{x[1+x^2]}{(1+x^2)(1+y^2)\sqrt{1+x^2+y^2}}$$

$$\frac{\partial v}{\partial y} = \frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}$$

Again PD wrt to x

$$\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = \frac{\partial}{\partial x} \left[\frac{x}{(1+y^2)\sqrt{1+x^2+y^2}} \right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{1}{(1+y^2)} \cdot \frac{\sqrt{1+x^2+y^2} \frac{\partial}{\partial x}(x) - x \frac{\partial}{\partial x}(\sqrt{1+x^2+y^2})}{(\sqrt{1+x^2+y^2})^2}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{1}{(1+y^2)} \cdot \frac{\sqrt{1+x^2+y^2}(1) - x \frac{1}{2\sqrt{1+x^2+y^2}} \cdot \frac{\partial}{\partial x}(1+x^2+y^2)}{(1+x^2+y^2)}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{1}{(1+y^2)} \cdot \frac{\sqrt{1+x^2+y^2} - x \frac{1}{2\sqrt{1+x^2+y^2}} \cdot (0+x+0)}{(1+x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{1+y^2} \cdot \frac{\sqrt{1+x^2+y^2} - \frac{x^2}{\sqrt{1+x^2+y^2}}}{1+x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{1+y^2} \cdot \frac{(\sqrt{1+x^2+y^2})^2 - x^2}{\sqrt{1+x^2+y^2} (1+x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{1+y^2} \cdot \frac{1+x^2+y^2 - x^2}{\sqrt{1+x^2+y^2}} \cdot \frac{1}{1+x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1+y^2}{(1+y^2) \sqrt{1+x^2+y^2} (1+x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$$

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}}$$

Hence proved.

④ If $au+b = x+a^2y$ S.T $\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 1$

Given, $au+b = x+a^2y$

$$au = x+a^2y-b$$

$$u = \frac{x+a^2y-b}{a}$$

P.D w.r.to x

$$\frac{\partial u}{\partial x} = \frac{1}{a} \cdot \frac{\partial}{\partial x} (x+a^2y-b)$$

$$= \frac{1}{a} (1+0+0)$$

$$\frac{\partial u}{\partial x} = \frac{1}{a} \quad \text{--- ①}$$

$\therefore u = \frac{x+a^2y-b}{a}$ P.D w.r.to y

$$\frac{\partial u}{\partial y} = \frac{1}{a} \cdot \frac{\partial}{\partial y} (x+a^2y-b)$$

$$= \frac{1}{a} (0+a^2(1)-0)$$

$$= \frac{a^2}{a}$$

$$\frac{\partial u}{\partial y} = a \quad \text{--- ②}$$

Equating eq ①, ②

$$\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = \frac{1}{a} \cdot a$$

$$\therefore \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 1$$

Hence proved

Q If $a^2x^2 + b^2y^2 - c^2z^2 = 0$. Then show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{a^2b^2(x^2+y^2)}{c^4z^3}$$

Given, $a^2x^2 + b^2y^2 - c^2z^2 = 0$ — (1)

Here z is a function
in terms of x & y

$$\times [a^2x^2 + b^2y^2 = c^2z^2]$$

$$z^2 = \frac{a^2x^2 + b^2y^2}{c^2}$$

$$a^2x^2 + b^2y^2 - c^2z^2 = 0$$

P.D w.r.to x

$$a^2 \cdot \frac{\partial}{\partial x}(x^2) + 0 - c^2 \cdot \frac{\partial}{\partial x}(z^2) = 0$$

$$a^2(2x) + 0 - c^2 \cdot 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$2a^2x = 2c^2z \cdot \frac{\partial z}{\partial x}$$

$$c^2z \cdot \frac{\partial z}{\partial x} = a^2x$$

$$z \cdot \frac{\partial z}{\partial x} = \frac{a^2x}{c^2} \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial x} = \frac{a^2x}{c^2 \cdot z}$$

$$z \cdot \frac{\partial z}{\partial x} = \frac{a^2 x}{c^2} \quad \text{--- (2)}$$

u v Again p.d w.r.to x.

$$z \cdot \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) + \frac{\partial z}{\partial x} \cdot \frac{\partial}{\partial x} (z) = \frac{a^2}{c^2} \cdot \frac{\partial}{\partial x} (x)$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x} = \frac{a^2}{c^2} \quad (1)$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x} \right)^2 = \frac{a^2}{c^2}$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} - \left(\frac{\partial z}{\partial x} \right)^2$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} - \left(\frac{a^2 x}{c^2 z} \right)^2 \quad [\text{from eq (2)}]$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} - \frac{a^4 x^2}{c^4 z^2}$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^4 z^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^4 z^2 \cdot z}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^4 \cdot z^3} \quad (3)$$

Similarly,

$$\frac{\partial^2 z}{\partial y^2} = \frac{b^2 c^2 z^2 - b^4 y^2}{c^4 z^3} \quad (4)$$

Adding eq (3) & (4)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^4 \cdot z^3} + \frac{b^2 c^2 z^2 - b^4 y^2}{c^4 \cdot z^3}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{a^2 c^2 z^2 - a^4 x^2 + b^2 c^2 z^2 - b^4 y^2}{c^4 z^3}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{c^2 z^2 (a^2 + b^2) - a^4 x^2 - b^4 y^2}{c^4 z^3}$$

[from eq
 $a^2 x^2 + b^2 y^2 = c^2 z^2$]

$$= \frac{(a^2 x^2 + b^2 y^2) (a^2 + b^2) - a^4 x^2 - b^4 y^2}{c^4 z^3}$$

$$= \frac{a^4 \cancel{x^2} + a^2 b^2 x^2 + a^2 b^2 y^2 + b^4 \cancel{y^2} - a^4 \cancel{x^2} - b^4 \cancel{y^2}}{c^4 z^3}$$

$$= \frac{a^2 b^2 (x^2 + y^2)}{c^4 z^3}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{a^2 b^2 (x^2 + y^2)}{c^4 z^3} //$$

Hence proved

② find the value of $\frac{1}{a^2} \cdot \frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2} \cdot \frac{\partial^2 z}{\partial y^2}$

when $a^2 x^2 + b^2 y^2 - c^2 z^2 = 0$

Given, $a^2 x^2 + b^2 y^2 - c^2 z^2 = 0$ — (1)

P.D w.r.to x

$$a^2(2x) + 0 - c^2 \cdot 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$2a^2 x = 2c^2 z \cdot \frac{\partial z}{\partial x}$$

$$z \cdot \frac{\partial z}{\partial x} = \frac{a^2 x}{c^2} \quad \text{--- (2)}$$

$$\left[\frac{\partial z}{\partial x} = \frac{a^2 x}{c^2 z} \right]$$

$$z \cdot \frac{\partial z}{\partial x} = \frac{a^2 x}{c^2}$$

Again P.D w.r.to x

$$z \cdot \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) + \frac{\partial z}{\partial x} \cdot \frac{\partial}{\partial x} (z) = \frac{a^2}{c^2} \cdot \frac{\partial}{\partial x} (x)$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x} = \frac{a^2}{c^2} \quad (1)$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} - \left(\frac{\partial z}{\partial x} \right)^2$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} - \left(\frac{a^2 x}{c^2 z} \right)^2 \quad [\text{from eq (2)}]$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} - \frac{a^4 x^2}{c^4 z^2}$$

$$z \cdot \frac{\partial^2 z}{\partial x^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^4 z^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^4 z^2 \cdot z}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^4 z^3}$$

both sides multiply with $\frac{1}{a^2}$

$$\frac{1}{a^2} \cdot \frac{\partial^2 z}{\partial x^2} = \frac{1}{\cancel{a^2}} \cdot \frac{\cancel{a^2} (c^2 z^2 - a^2 x^2)}{c^4 z^3}$$

$$\frac{1}{a^2} \cdot \frac{\partial^2 z}{\partial x^2} = \frac{c^2 z^2 - a^2 x^2}{c^4 z^3} \quad \text{--- (3)}$$

Similarly,

$$\frac{1}{b^2} \cdot \frac{\partial^2 z}{\partial y^2} = \frac{c^2 z^2 - b^2 y^2}{c^4 z^3} \quad \text{--- (4)}$$

Adding eq ③ & ④

$$\frac{1}{a^2} \cdot \frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2} \cdot \frac{\partial^2 z}{\partial y^2} = \frac{c^2 z^2 - a^2 x^2}{c^4 z^3} + \frac{c^2 z^2 - b^2 y^2}{c^4 z^3}$$

$$= \frac{c^2 z^2 - a^2 x^2 + c^2 z^2 - b^2 y^2}{c^4 z^3}$$

$$= \frac{c^2 z^2 - [a^2 x^2 - c^2 z^2 + b^2 y^2]}{c^4 z^3}$$

$$= \frac{c^2 z^2 - [a^2 x^2 + b^2 y^2 - c^2 z^2]}{c^4 z^3}$$

$$= \frac{c^2 z^2 - [0]}{c^4 z^3} \quad [\text{from eq ①}]$$

$$= \frac{c^2 z^2}{c^4 z^3}$$

$$= \frac{1}{c^2 z}$$

$$\therefore \frac{1}{a^2} \cdot \frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2} \cdot \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}$$

Example: If $x^x y^y z^z = c$ at $x=y=z$

Show that $\frac{\partial^2 z}{\partial x \partial y} = -[x \log ex]^{-1}$

Given, $x^x y^y z^z = c$

Here z is a function
in terms of x and y

Apply log on b.s.

$$\log [x^x y^y z^z] = \log c$$

$$[\log (a^b) = b \log a + \log b]$$

$$[\log a^m = m \log a]$$

$$\log x^x + \log y^y + \log z^z = \log c$$

$$x \log x + y \log y + z \log z = \log c$$

P.D w.r.to y

$$x \left[0 + y \cdot \frac{\partial}{\partial y} (\log y) + \log y \cdot \frac{\partial}{\partial y} (y) + \cancel{\log c} \right] \times$$

$$0 + y \cdot \frac{\partial}{\partial y} (\log y) + \log y \cdot \frac{\partial}{\partial y} (y) + z \cdot \frac{\partial}{\partial y} (\log z) + \log z \cdot \frac{\partial}{\partial y} (z) = 0$$

$$y \cdot \frac{1}{y} + \log y (1) + z \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial y} + \log z \cdot \frac{\partial z}{\partial y} = 0$$

$$1 + \log y + \frac{\partial z}{\partial y} + \log z \cdot \frac{\partial z}{\partial y} = 0$$

$$(1 + \log y) + \frac{\partial z}{\partial y} (1 + \log z) = 0$$

$$(1+\log y) + \frac{\partial z}{\partial y} (1+\log z) = 0$$

$$\frac{\partial z}{\partial y} (1+\log z) = -(1+\log y)$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z}$$

Similarly,

$$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{1+\log z}$$

P.D wrto x

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left[\frac{-(1+\log y)}{1+\log z} \right]$$

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y) \cdot \frac{\partial}{\partial x} \left(\frac{1}{1+\log z} \right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y) \cdot \frac{-1}{(1+\log z)^2} \cdot \frac{\partial}{\partial x} (1+\log z)$$

$$= -(1+\log y) \cdot \frac{-1}{(1+\log z)^2} \cdot \left[0 + \frac{1}{z} \cdot \frac{\partial z}{\partial x} \right]$$

$$= \frac{(1+\log y)}{(1+\log z)^2} \cdot \left[0 + \frac{1}{z} \cdot \frac{\partial z}{\partial x} \right]$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log y)}{(1+\log z)^2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}$$

$$= \frac{1+\log y}{(1+\log z)^2} \cdot \frac{1}{z} \left[\frac{-(1+\log x)}{(1+\log z)} \right]$$

$$= \frac{-(1+\log y)(1+\log x)}{(1+\log z)^3 \cdot z}$$

given, at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\log x)(1+\log x)}{(1+\log x)^3 \cdot x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{(1+\log x) \cdot x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{(\log e + \log x) x}$$

$$= \frac{-1}{\log(ex) \cdot x}$$

$$= \frac{-1}{x \log(ex)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left[-x \log(ex) \right]^{-1} //$$

Hence proved //

HOMOGENEOUS FUNCTION

If $f(kx, ky) = k^n f(x, y)$ then $f(x, y)$ is called Homogeneous function of degree n .

Here, n is called degree of the homogeneous equation.

$$(i) f(x, y) = \frac{x^4 + y^4}{3xy}$$

$$f(kx, ky) = \frac{(kx)^4 + (ky)^4}{3(kx)(ky)}$$

$$f(kx, ky) = \frac{k^4 x^4 + k^4 y^4}{3k^2 xy}$$

$$= \frac{k^4 [x^4 + y^4]}{k^2 (3xy)}$$

$$= k^2 \frac{x^4 + y^4}{3xy}$$

$$f(kx, ky) = k^2 f(x, y)$$

f is a Homogeneous function of degree 2.

$$(2) \quad f(x, y) = \frac{x^2 + y^2}{x^5 - y^5}$$

$$f(kx, ky) = \frac{(kx)^2 + (ky)^2}{(kx)^5 - (ky)^5}$$

$$= \frac{k^2 x^2 + k^2 y^2}{k^5 x^5 - k^5 y^5}$$

$$= \frac{k^2 [x^2 + y^2]}{k^5 [x^5 - y^5]}$$

$$= k^{2-5} \left[\frac{x^2 + y^2}{x^5 - y^5} \right]$$

$$f(kx, ky) = k^{-3} f(x, y)$$

$\therefore f$ is a homogenous function of degree -3

$$(3) \quad f(x, y) = \frac{x+y}{\sqrt{x} + \sqrt{y}}$$

$$f(kx, ky) = \frac{kx + ky}{\sqrt{kx} + \sqrt{ky}}$$

$$= \frac{k(x+y)}{\sqrt{k}(\sqrt{x} + \sqrt{y})}$$

$$= \frac{k^1}{k^{1/2}} \frac{(x+y)}{(\sqrt{x} + \sqrt{y})}$$

$$= k^{1-\frac{1}{2}} f(x, y)$$

$$= k^{\frac{1}{2}} f(x, y)$$

$\therefore f$ is a homogenous function of degree $\frac{1}{2}$.

$$(iv) f(x, y) = \sin^{-1}\left(\frac{x^2}{y^2}\right)$$

$$f(kx, ky) = \sin^{-1}\left(\frac{k^2 x^2}{k^2 y^2}\right)$$

$$= \sin^{-1}\left(\frac{x^2}{y^2}\right)$$

$$f(kx, ky) = f(x, y)$$

f is a homogenous function of degree 0.

$$(v) f(x, y) = \tan\left(\frac{x^2}{y^2}\right)$$

$$f(kx, ky) = \tan\left(\frac{k^2 x^2}{k^2 y^2}\right)$$

$$= \tan\left(k \cdot \frac{x^2}{y^2}\right)$$

$$x_0 \left\{ \begin{array}{l} = k \tan\left(\frac{x^2}{y^2}\right) \\ = k f \end{array} \right\} x_0$$

$$\neq f(x, y)$$

f is a non-homogenous function.

Euler's Theorem:

If z is a homogenous function of degree n

$$\text{Then } x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz.$$

Verify the Euler's Theorem:

$$z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

Given,

$$z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$z(kx, ky) = \sin^{-1}\left(\frac{kx}{ky}\right) + \tan^{-1}\left(\frac{ky}{kx}\right)$$

$$= \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$z(kx, ky) = z$$

$\therefore z$ is a homogenous function of degree 0

By the Euler's Theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz.$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 0 \cdot z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 0 \quad \text{--- (1)}$$

$$z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

P.D. w.r.to x

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{\partial \left(\frac{x}{y}\right)}{\partial x} + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial \left(\frac{y}{x}\right)}{\partial x}$$

$$= \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{\partial \left(\frac{x}{y}\right)}{\partial x} + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial \left(\frac{y}{x}\right)}{\partial x}$$

$$= \frac{1}{\sqrt{\frac{y^2 - x^2}{y^2}}} \cdot \frac{1}{y} (1) + \frac{1}{\frac{x^2 + y^2}{x^2}} \cdot y \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{\frac{\sqrt{y^2 - x^2}}{\sqrt{y^2}}} \cdot \frac{1}{y} + \frac{1}{\frac{x^2 + y^2}{x^2}} \cdot \frac{-y}{x^2}$$

$$= \frac{y}{\sqrt{y^2 - x^2}} \cdot \frac{1}{y} + \frac{x^2}{x^2 + y^2} \cdot \frac{-y}{x^2}$$

$$= \frac{1}{\sqrt{y^2 - x^2}} + \frac{-y}{x^2 + y^2}$$

$$= \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

both sides multiply with x

$$\frac{\partial z}{\partial x} \cdot x = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \quad \text{--- (2)}$$

$$z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

PD w.r.to y

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial y} \left(\frac{x}{y}\right) + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x}\right)$$

$$= \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot x \cdot \left(\frac{-1}{y^2}\right) + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} \cdot (1)$$

$$= \frac{1}{\sqrt{\frac{y^2 - x^2}{y^2}}} \cdot \left(\frac{-x}{y^2}\right) + \frac{1}{\frac{x^2 + y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{y}{\sqrt{y^2 - x^2}} \cdot \frac{-x}{y^2} + \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x}$$

$$= \frac{-x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

Multiplying both sides by

$$y \cdot \frac{\partial z}{\partial y} = \frac{-xy}{y\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$y \cdot \frac{\partial z}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{--- (3)}$$

Adding eq (2) & (3)

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 0 \quad \text{--- (4)}$$

from (1) & (4)

Euler's Theorem is verified.

Verify Euler's Theorem of

$$Z = ax^2 + 2hxy + by^2$$

Given, $Z = ax^2 + 2hxy + by^2$

$$z(kx, ky) = a(kx)^2 + 2h(kx)(ky) + b(ky)^2$$

$$= a k^2 x^2 + 2h k^2 xy + b k^2 y^2$$

$$= k^2 [ax^2 + 2hxy + by^2]$$

$$z(kx, ky) = k^2 z$$

z is a homogenous function of degree 2

By the Euler's Theorem,

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2z \quad \text{--- ①}$$

$$z = ax^2 + 2hxy + by^2$$

pp w.r.to x

$$\frac{\partial z}{\partial x} = a(2x) + 2hy(1) + 0$$

$$\frac{\partial z}{\partial x} = 2ax + 2hy$$

Multiplying both sides with x

$$x \cdot \frac{\partial z}{\partial x} = x [2ax + 2hy]$$

$$x \cdot \frac{\partial z}{\partial x} = 2ax^2 + 2hxy \quad \text{--- ②}$$

$$z = ax^2 + 2hxy + by^2$$

P.D. w.r.to y

$$\frac{\partial z}{\partial y} = 0 + 2hx(1) + b(2y)$$

$$\frac{\partial z}{\partial y} = 2hx + 2by$$

multiplying both sides with y

$$y \cdot \frac{\partial z}{\partial y} = y(2hx + 2by)$$

$$y \cdot \frac{\partial z}{\partial y} = 2hxy + 2by^2 \quad \text{--- (3)}$$

Adding eq ① & ②

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y}$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2ax^2 + 4hxy + 2by^2$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2[ax^2 + 2hxy + by^2]$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2z \quad \text{--- (4)}$$

from eq ① & ④

Euler's theorem is verified.

$$\text{i.e., } x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2z //$$

Verify the Euler's theorem to the function

$$z = (x^2 + xy + y^2)^{-1}$$

$$\text{Given } z = (x^2 + xy + y^2)^{-1}$$

$$z(kx, ky) = [(kx)^2 + (kx)(ky) + (ky)^2]^{-1}$$

$$= [k^2 x^2 + k^2 xy + k^2 y^2]^{-1}$$

$$= [k^2 [x^2 + xy + y^2]]^{-1}$$

$$= (k^2)^{-1} \underbrace{(x^2 + xy + y^2)^{-1}}$$

$$z(kx, ky) = k^{-2} \cdot z$$

Here z is Homogenous function of degree -2

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = (-2)z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = -2z \quad \text{--- (1)}$$

$$z = (x^2 + xy + y^2)^{-1}$$

P.D wrto x

$$\frac{\partial z}{\partial x} = (-1) (x^2 + xy + y^2)^{-1-1} \cdot \frac{\partial}{\partial x} (x^2 + xy + y^2)$$

$$= (-1) (x^2 + xy + y^2)^{-2} \cdot [2x + y(1) + 0]$$

$$= (-1) (x^2 + xy + y^2)^{-2} (2x + y)$$

$$\frac{\partial z}{\partial x} = -(2x + y) (x^2 + xy + y^2)^{-2}$$

B.S multiply with x

$$x \cdot \frac{\partial z}{\partial x} = -(2x^2 + xy) (x^2 + xy + y^2)^{-2} \quad \text{--- (2)}$$

$$z = (x^2 + xy + y^2)^{-1}$$

P.D wrto y

$$\frac{\partial z}{\partial y} = -1 (x^2 + xy + y^2)^{-1-1} \cdot \frac{\partial}{\partial y} (x^2 + xy + y^2)$$

$$\frac{\partial z}{\partial y} = (-1)(x^2 + xy + y^2)^{-2} (0 + x(1) + 2y)$$

$$\frac{\partial z}{\partial y} = (-1)(x^2 + xy + y^2)^{-2} (x + 2y)$$

$$\frac{\partial z}{\partial y} = -(x + 2y)(x^2 + xy + y^2)^{-2}$$

Now multiply with y

$$y \cdot \frac{\partial z}{\partial y} = -(xy + 2y^2)(x^2 + xy + y^2)^{-2} \quad \text{--- (3)}$$

Adding eq (2) & (3)

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{-(2x^2 + xy)(x^2 + xy + y^2)^{-2} + -(xy + 2y^2)(x^2 + xy + y^2)^{-2}}{(x^2 + xy + y^2)^2}$$

$$= (x^2 + xy + y^2)^{-2} [-2x^2 - xy - xy - 2y^2]$$

$$= (x^2 + xy + y^2)^{-2} [-2x^2 - 2y^2 - 2xy]$$

$$= (x^2 + xy + y^2)^{-2} \cdot -2 [x^2 + y^2 + xy]$$

$$= -2 (x^2 + xy + y^2)^{-2+1}$$

$$= -2 (x^2 + xy + y^2)^{-1}$$

$$= -2z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = -2z \quad \text{--- (4)}$$

from eq ① & ④

Euler's theorem is verified //

$$\textcircled{*} \quad z = x^n \log\left(\frac{y}{x}\right)$$

Given, $z = x^n \log\left(\frac{y}{x}\right)$

$$z(kx, ky) = (kx)^n \log\left(\frac{ky}{kx}\right)$$

$$z(kx, ky) = k^n x^n \log\left(\frac{y}{x}\right)$$

$$z(kx, ky) = k^n \cdot z$$

Here z is a homogenous function of degree n .

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz \quad \text{--- (1)}$$

$$z = x^n \log\left(\frac{y}{x}\right)$$

P.D w.r.to x

$$\frac{\partial z}{\partial x} = x^n \frac{\partial}{\partial x} \left(\log\left(\frac{y}{x}\right) \right) + \log\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial x} (x^n)$$

$$\frac{\partial z}{\partial x} = x^n \cdot \frac{1}{(y/x)} \cdot \frac{\partial}{\partial x} (y/x) + \log(y/x) \cdot n \cdot x^{n-1}$$

$$\frac{\partial z}{\partial x} = x^n \cdot \left(\frac{x}{y}\right) \cdot y \cdot \left(\frac{-1}{x^2}\right) + \log(y/x) \cdot n \cdot x^{n-1}$$

$$\frac{\partial z}{\partial x} = x^n \left(\frac{x}{y}\right) \left(\frac{-y}{x^2}\right) + \log\left(\frac{y}{x}\right) \cdot n x^{n-1}$$

$$\frac{\partial z}{\partial x} = x^n \left(\frac{-1}{x}\right) + \log\left(\frac{y}{x}\right) n x^{n-1}$$

$$= -x^n \cdot x^{-1} + \log\left(\frac{y}{x}\right) n x^{n-1}$$

$$\frac{\partial z}{\partial x} = -x^{n-1} + \log\left(\frac{y}{x}\right) n x^{n-1} \quad \text{--- (2)} \Rightarrow \text{B.S multiply with } x$$

$$z = x^n \log\left(\frac{y}{x}\right)$$

P.D w.r.to y

$$\frac{\partial z}{\partial y} = x^n \cdot \frac{\partial}{\partial y} (\log(y/x))$$

$$\frac{\partial z}{\partial y} = x^n \cdot \frac{1}{y/x} \cdot \frac{\partial}{\partial y} (y/x)$$

$$\frac{\partial z}{\partial y} = x^n \cdot \frac{x}{y} \cdot \frac{1}{x} \cdot (1)$$

$$\frac{\partial z}{\partial y} = x^n \cdot \frac{1}{y} \quad \text{--- (3)}$$

eq ② multiply with x

$$x \cdot \frac{\partial z}{\partial x} = -x^{n-1} \cdot x + \log\left(\frac{y}{x}\right) \cdot n \cdot x^{n-1} \cdot x$$

$$x \cdot \frac{\partial z}{\partial x} = -x^{n-1+1} + \log\left(\frac{y}{x}\right) \cdot n \cdot x^{n-1+1}$$

$$x \cdot \frac{\partial z}{\partial x} = -x^n + \log\left(\frac{y}{x}\right) \cdot n \cdot x^n \text{ --- ④}$$

eq ③ multiply with y

$$y \cdot \frac{\partial z}{\partial y} = y \cdot x^n \cdot \frac{1}{y}$$

$$y \cdot \frac{\partial z}{\partial y} = x^n \text{ --- ⑤}$$

Adding eq ④ & ⑤

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = -x^n + \log\left(\frac{y}{x}\right) \cdot n \cdot x^n + x^n$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n \cdot x^n \cdot \log\left(\frac{y}{x}\right)$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n z \text{ --- ⑥}$$

from eq ① & ⑥

Euler's theorem is verified //

$$⑧ \quad z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

Given,

$$z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

$$z(kx, ky) = \frac{(kx)^{1/4} + (ky)^{1/4}}{(kx)^{1/5} + (ky)^{1/5}}$$

$$z(kx, ky) = \frac{k^{1/4} x^{1/4} + k^{1/4} y^{1/4}}{k^{1/5} x^{1/5} + k^{1/5} y^{1/5}}$$

$$z(kx, ky) = \frac{k^{1/4} [x^{1/4} + y^{1/4}]}{k^{1/5} [x^{1/5} + y^{1/5}]}$$

$$z(kx, ky) = k^{1/4 - 1/5} \cdot z$$

$$z(kx, ky) = k^{\frac{5-4}{20}} z$$

$$z(kx, ky) = k^{1/20} z$$

Here z is a Homogenous function of degree $\frac{1}{20}$

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \lambda z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{1}{20} \cdot z \quad \text{--- ①}$$

$$z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

P.D wrto x

$$\frac{\partial z}{\partial x} = \frac{x^{1/5} + y^{1/5} \cdot \frac{\partial}{\partial x}(x^{1/4} + y^{1/4}) - (x^{1/4} + y^{1/4}) \cdot \frac{\partial}{\partial x}(x^{1/5} + y^{1/5})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{x^{1/5} + y^{1/5} \cdot \left[\frac{1}{4} x^{-3/4} + 0 \right] - (x^{1/4} + y^{1/4}) \left[\frac{1}{5} x^{-4/5} + 0 \right]}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{x^{1/5} + y^{1/5} \left[\frac{1}{4} x^{-3/4} \right] - (x^{1/4} + y^{1/4}) \left[\frac{1}{5} x^{-4/5} \right]}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{[x^{1/5} + y^{1/5}] \left[\frac{1}{4} x^{-3/4} \right] - [x^{1/4} + y^{1/4}] \left[\frac{1}{5} x^{-4/5} \right]}{(x^{1/5} + y^{1/5})^2}$$

B.S Multiply with x

$$9. \frac{\partial z}{\partial x} = \frac{x [x^{1/5} + y^{1/5}] [\frac{1}{4} x^{-3/4}] - x [x^{1/4} + y^{1/4}] [\frac{1}{5} x^{-4/5}]}{(x^{1/5} + y^{1/5})^2}$$

$$10. \frac{\partial z}{\partial x} = \frac{[x^{1/5} + y^{1/5}] [\frac{1}{4} x^{-3/4+1}] - [x^{1/4} + y^{1/4}] [\frac{1}{5} x^{-4/5+1}]}{(x^{1/5} + y^{1/5})^2}$$

$$11. \frac{\partial z}{\partial x} = \frac{(x^{1/5} + y^{1/5}) (\frac{1}{4} x^{1/4}) - (x^{1/4} + y^{1/4}) (\frac{1}{5} x^{1/5})}{(x^{1/5} + y^{1/5})^2} \quad \text{--- (2)}$$

$$Z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

p.d w.r.to y

$$\frac{\partial z}{\partial y} = \frac{x^{1/5} + y^{1/5} \cdot \frac{\partial}{\partial y} (x^{1/4} + y^{1/4}) + (x^{1/4} + y^{1/4}) \frac{\partial}{\partial y} (x^{1/5} + y^{1/5})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{x^{1/5} + y^{1/5} \cdot (0 + \frac{1}{4} y^{1/4-1}) + (x^{1/4} + y^{1/4}) (0 + \frac{1}{5} y^{1/5-1})}{(x^{1/5} + y^{1/5})^2}$$

$$\frac{\partial z}{\partial y} = \frac{x^{1/5} + y^{1/5} (\frac{1}{4} y^{-3/4}) + (x^{1/4} + y^{1/4}) (\frac{1}{5} y^{-4/5})}{(x^{1/5} + y^{1/5})^2}$$

B.S multiply with y

$$y \cdot \frac{\partial z}{\partial y} = \frac{y (x^{1/5} + y^{1/5}) \left(\frac{1}{4} y^{-3/4} \right) - (x^{1/4} + y^{1/4}) y \left(\frac{1}{5} y^{-4/5} \right)}{(x^{1/5} + y^{1/5})^2}$$

$$y \cdot \frac{\partial z}{\partial y} = \frac{x^{1/5} + y^{1/5} \left(\frac{1}{4} y^{-3/4+1} \right) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} y^{-4/5+1} \right)}{(x^{1/5} + y^{1/5})^2}$$

$$y \cdot \frac{\partial z}{\partial y} = \frac{(x^{1/5} + y^{1/5}) \left(\frac{1}{4} y^{1/4} \right) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} y^{1/5} \right)}{(x^{1/5} + y^{1/5})^2} \quad \text{--- (3)}$$

Adding eq (2) & (3)

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y}$$

$$= \frac{x^{1/5} + y^{1/5} \left(\frac{1}{4} x^{1/4} \right) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} x^{1/5} \right)}{(x^{1/5} + y^{1/5})^2} + \frac{x^{1/5} + y^{1/5} \left(\frac{1}{4} y^{1/4} \right) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} y^{1/5} \right)}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{x^{1/5} + y^{1/5} \left(\frac{1}{4} x^{1/4} \right) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} x^{1/5} \right) + x^{1/5} + y^{1/5} \left(\frac{1}{4} y^{1/4} \right) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} y^{1/5} \right)}{(x^{1/5} + y^{1/5})^2}$$

$$\frac{x^{1/5} + y^{1/5} \left[\frac{1}{4} x^{1/4} + \frac{1}{4} y^{1/4} \right] - (x^{1/4} + y^{1/4}) \left[\frac{1}{5} x^{1/5} + \frac{1}{5} y^{1/5} \right]}{(x^{1/5} + y^{1/5})^2}$$

$$\frac{x^{1/5} + y^{1/5} \left(\frac{1}{4} \right) (x^{1/4} + y^{1/4}) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} \right) (x^{1/5} + y^{1/5})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{\cancel{x^{1/5} + y^{1/5}} \left[\frac{1}{4} x^{1/4} + y^{1/4} - \frac{1}{5} x^{1/4} - y^{1/4} \right]}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{x^{1/4} + y^{1/4} \left[\frac{1}{4} - \frac{1}{5} \right]}{x^{1/5} + y^{1/5}}$$

$$= \left(\frac{1}{4} - \frac{1}{5} \right) z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \left(\frac{5-4}{20} \right) z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \left(\frac{1}{20} \right) z \quad \text{--- (4)}$$

from eq (1) & (4)

Euler's theorem is verified. //

④ If $z \cdot (x+y) = x^2 + y^2$ s.t. $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4 \left[1 - \frac{\partial z}{\partial x}\right]$

Given,

$$z(x+y) = x^2 + y^2$$

$$z = \frac{x^2 + y^2}{x+y}$$

P.D wrto x

$$\frac{\partial z}{\partial x} = \frac{x+y \cdot \frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2) \cdot \frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x+y[2x+0] - (x^2+y^2)[1+0]}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2 + 2xy - y^2}{(x+y)^2} \quad \text{--- ①}$$

$$z = \frac{x^2 + y^2}{x+y}$$

P.D wrto y

$$\frac{\partial z}{\partial y} = \frac{x+y \cdot \frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2) \cdot \frac{\partial}{\partial y}(x+y)}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{x+y[0+2y] - (x^2+y^2)[0+1]}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2xy + 2y^2 - x^2 - y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x+y)^2} \quad \text{--- (2)}$$

subtracting eq (1) & (2)

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

$$= \frac{x^2 + \cancel{2xy} - y^2 - y^2 - \cancel{2xy} + x^2}{(x+y)^2}$$

$$= \frac{2x^2 - 2y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x^2 - y^2)}{(x+y)^2}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = \left[\frac{2 \cdot (x^2 - y^2)}{(x+y)^2} \right]^2$$

$$= 4 \frac{(x^2 - y^2)^2}{(x+y)^4}$$

$$= 4 \frac{[(x+y)(x-y)]^2}{(x+y)^4}$$

$$= 4 \frac{(x+y)^2 (x-y)^2}{(x+y)^4}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = 4 \cdot \frac{(x-y)^2}{(x+y)^2} \quad \text{--- (3)}$$

Now,

$$\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right] = \left[1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2} \right]$$

$$= \frac{(x+y)^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}$$

$$= \frac{x^2 + y^2 + 2xy - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}$$

$$= \frac{x^2 + y^2 - 2xy}{(x+y)^2}$$

$$= \frac{(x-y)^2}{(x+y)^2}$$

$$\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right] = \frac{(x-y)^2}{(x+y)^2}$$

multiply 4 on b.s

$$4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right] = 4 \frac{(x-y)^2}{(x+y)^2} \quad \text{--- (4)}$$

equating eq (3) & (4)

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = 4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right] //$$

hence proved.

* find the degree of

$$z = x^n + \left(\frac{y}{x} \right)$$

$$z(kx, ky) = (kx)^n + \left[\frac{ky}{kx} \right]$$

$$= (kx)^n + \left(\frac{y}{x} \right)$$

$$= k^n x^n + \left(\frac{y}{x} \right)$$

$$z(kx, ky) = k^n x^n \cdot f\left(\frac{y}{x}\right) \\ = k^n f\left(\frac{y}{x}\right) \cdot x^n$$

$$\boxed{z(kx, ky) = k^n \cdot z}$$

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mp

Euler's

State and Prove that Euler's theorem

Statement: If $z = f(x, y)$ is a homogenous function in terms of x and y of degree n then

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n z$$

Proof: let $z = x^n \cdot f\left(\frac{y}{x}\right)$

P.D. w.r.to x

$$\frac{\partial z}{\partial x} = x^n \cdot \frac{\partial}{\partial x} f\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial x} (x^n)$$

$$= x^n \cdot f'\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial x} \left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) \cdot n \cdot x^{n-1}$$

$$= x^n \cdot f'\left(\frac{y}{x}\right) y \cdot \frac{\partial}{\partial x} \left(\frac{1}{x}\right) + f\left(\frac{y}{x}\right) \cdot n \cdot x^{n-1}$$

$$= x^n \cdot f'\left(\frac{y}{x}\right) \cdot y \left(-\frac{1}{x^2}\right) + f\left(\frac{y}{x}\right) \cdot n \cdot x^{n-1}$$

$$= x^n \cdot f'\left(\frac{y}{x}\right) y \cdot x^{-2} + f\left(\frac{y}{x}\right) n \cdot x^{n-1}$$

$$\frac{\partial z}{\partial x} = -x^{n-2} \cdot f'(y/x) y + n \cdot x^{n-1} \cdot f(y/x)$$

both sides multiply with x

$$x \cdot \frac{\partial z}{\partial x} = x \left[-x^{n-2} \cdot f'(y/x) y + n x^{n-1} f(y/x) \right]$$

$$x \cdot \frac{\partial z}{\partial x} = -x \cdot x^{n-2} \cdot f'(y/x) y + n \cdot x \cdot x^{n-1} \cdot f(y/x)$$

$$= -x^{1+n-2} f'(y/x) y + n \cdot x^{1+n-1} \cdot f(y/x)$$

$$x \cdot \frac{\partial z}{\partial x} = -x^{n-1} \cdot f'(y/x) \cdot y + n \cdot x^n \cdot f(y/x) \quad \text{--- (1)}$$

$$z = x^n \cdot f(y/x)$$

PD w.r.to y

$$\frac{\partial z}{\partial y} = x^n \cdot \frac{\partial}{\partial y} f(y/x)$$

$$= x^n \cdot f'(y/x) \cdot \frac{\partial}{\partial y} (y/x)$$

$$= x^n \cdot f'(y/x) \cdot \frac{1}{x} \quad (1)$$

$$= x^n \cdot f'(y/x) \cdot x^{-1}$$

$$\frac{\partial z}{\partial y} = x^{n-1} f'(y/x)$$

Multiply with y on bs

$$y \cdot \frac{\partial z}{\partial y} = y \cdot x^{n-1} \cdot f'(y/x) \quad \text{--- (2)}$$

Adding eq (1) & (2)

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = -x^{n-1} f'(y/x) y + n \cdot x^n f'(y/x) + y \cdot x^{n-1} f'(y/x)$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n \cdot x^n f'(y/x)$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n \cdot z$$

Hence proved.

(b) If $z = xy \cdot f(y/x)$ show that $x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2z$

Given

$$z = xy \cdot f(y/x)$$

$$z(kx, ky) = (kx)(ky) \cdot f\left(\frac{ky}{kx}\right)$$

$$= k^2 xy \cdot f(y/x)$$

$$z(kx, ky) = k^2 \cdot z$$

z is a homogenous function of degree 2

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2z$$

Hence proved.

* Statement. If z is a homogenous function in terms of x and y of degree n then

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy \cdot \frac{\partial^2 z}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Given that,

z is a homogenous function in terms of x and y of degree n .

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz \quad \text{--- (1)}$$

p.d w.r.to x

$$x \cdot \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) + \frac{\partial z}{\partial x} \frac{\partial (x)}{\partial x} + y \cdot \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = n \cdot \frac{\partial}{\partial x} (z)$$

$$x \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} (1) + y \cdot \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x}$$

$$x \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} (1) + y \cdot \frac{\partial^2 z}{\partial x \partial y} = n \cdot \frac{\partial z}{\partial x}$$

both sides multiply with x

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + x \cdot \frac{\partial z}{\partial x} + xy \cdot \frac{\partial^2 z}{\partial x \partial y} = nx \cdot \frac{\partial z}{\partial x}$$

— (2)

Similarly,

$$y^2 \cdot \frac{\partial^2 z}{\partial y^2} + y \cdot \frac{\partial z}{\partial y} + xy \cdot \frac{\partial^2 z}{\partial x \partial y} = ny \cdot \frac{\partial z}{\partial y} \quad \text{--- (3)}$$

Adding eq (2) and (3)

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + x \cdot \frac{\partial z}{\partial x} + xy \cdot \frac{\partial^2 z}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} + y \cdot \frac{\partial z}{\partial y} + xy \cdot \frac{\partial^2 z}{\partial x \partial y}$$

$$= nx \cdot \frac{\partial z}{\partial x} + ny \cdot \frac{\partial z}{\partial y}$$

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} + 2xy \cdot \frac{\partial^2 z}{\partial x \partial y} + x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y}$$

$$= n \left[x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right]$$

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} + 2xy \cdot \frac{\partial^2 z}{\partial x \partial y} + \underbrace{\left(x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right)}_{\text{from eq (1)}} = n \left[x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right]$$

From eq (1)

$$x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} + nz + 2xy \frac{\partial^2 z}{\partial x \partial y} = n(nz) \quad [\text{from eq.}]$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} = n(nz) - nz$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} = nz(n-1)$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Hence proved. //

NOTE:

If v is a homogenous function of degree n

Then $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also homogenous function

of degree $(n-1)$

Theorem: If U is a homogenous function of degree n then $x \cdot \frac{\partial^2 U}{\partial x^2} + y \cdot \frac{\partial^2 U}{\partial y \partial x} = (n-1) \frac{\partial U}{\partial x}$

Proof: Given U is homogenous function of degree n

$\frac{\partial U}{\partial x}$ is also homogenous function of degree $(n-1)$

By the Euler's Theorem.

$$x \cdot \frac{\partial}{\partial x} \left(\frac{\partial U}{\partial x} \right) + y \cdot \frac{\partial}{\partial y} \left(\frac{\partial U}{\partial x} \right) = (n-1) \frac{\partial U}{\partial x}$$

$$x \cdot \frac{\partial^2 U}{\partial x^2} + y \cdot \frac{\partial^2 U}{\partial y \partial x} = (n-1) \frac{\partial U}{\partial x}$$

Hence proved

NOTE:

If U is a homogenous function of degree n

$$\text{then } y \cdot \frac{\partial^2 U}{\partial y^2} + x \cdot \frac{\partial^2 U}{\partial x \partial y} = (n-1) \frac{\partial U}{\partial y}$$

2019 12 marks
Q If $u(x, y) = \frac{y^3 - x^3}{y^2 + x^2}$ Then using Euler's Theorem

show that $x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = 0$

$$\text{Given, } u(x, y) = \frac{y^3 - x^3}{y^2 + x^2}$$

$$u(kx, ky) = \frac{(ky)^3 - (kx)^3}{(ky)^2 + (kx)^2}$$

$$= \frac{k^3 y^3 - k^3 x^3}{k^2 y^2 + k^2 x^2}$$

$$= \frac{k^3 (y^3 - x^3)}{k^2 (y^2 + x^2)}$$

$$= k^3 \cdot k^{-2} \cdot u(x, y)$$

$$u(kx, ky) = k u(x, y)$$

$\therefore u$ is a homogenous function of degree 1

By Euler's theorem
second order

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = n(n+1)u$$

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = 1(1+1)u$$

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence proved ✓

2019 (4m) V.V*

② If $U = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$ then show that

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \tan U$$

Given,

$$U = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$$

U is not a homogeneous function

$$U = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$$

$$\sin U = \frac{x^2 + y^2}{x + y}$$

$$\text{let, } z = \sin U = \frac{x^2 + y^2}{x + y}$$

$\therefore z$ is a homogeneous function of degree 1

By the Euler's Theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n \cdot z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n \cdot z$$

$$x \cdot \frac{\partial}{\partial x}(\sin u) + y \cdot \frac{\partial}{\partial y}(\sin u) = 1 \cdot \sin u$$

$$x \cdot \cos u \cdot \frac{\partial u}{\partial x} + y \cdot \cos u \cdot \frac{\partial u}{\partial y} = 1 \cdot \sin u$$

$$\cos u \left[x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} \right] = \sin u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \tan u$$

Hence proved //

Example:

③ If $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then show that

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + \frac{1}{4} \sin 2u = 0$$

Given, $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

u is not a Homogeneous function

$$V = \cot^{-1} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$$

$$\cot V = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\text{let, } z = \cot V = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$\therefore z$ is a homogenous function in terms of x and y
of degree $\frac{1}{2}$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n z$$

$$x \cdot \frac{\partial}{\partial x} (\cot V) + y \cdot \frac{\partial}{\partial y} (\cot V) = \frac{1}{2} (\cot V)$$

$$x \cdot (-\operatorname{cosec}^2 V) \cdot \frac{\partial V}{\partial x} + y \cdot (-\operatorname{cosec}^2 V) \cdot \frac{\partial V}{\partial y} = \frac{1}{2} (\cot V)$$

$$-\operatorname{cosec}^2 V \left[x \cdot \frac{\partial V}{\partial x} + y \cdot \frac{\partial V}{\partial y} \right] = \frac{1}{2} \cot V$$

$$x \cdot \frac{\partial V}{\partial x} + y \cdot \frac{\partial V}{\partial y} = -\frac{1}{2} \frac{\cot V}{\operatorname{cosec}^2 V}$$

$$x \cdot \frac{\partial V}{\partial x} + y \cdot \frac{\partial V}{\partial y} = -\frac{1}{2} \cot V \cdot \frac{1}{\operatorname{cosec}^2 V}$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = -\frac{1}{2} \cot u \cdot \sin^2 u$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = -\frac{1}{2} \frac{\cos u}{\sin u} \cdot \sin^2 u$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = -\frac{1}{2} \cos u \cdot \sin u$$

multiple and divide with 2

$$= -\frac{1}{2} \left[\frac{2 \cos u \sin u}{2} \right]$$

$$= -\frac{1}{4} \sin 2u$$

$$\boxed{2 \cos \theta \sin \theta = \sin 2\theta}$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = -\frac{1}{4} \sin 2u$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + \frac{1}{4} \sin 2u = 0$$

Hence proved.

$$\textcircled{+} \text{ If } u = \cos^{-1} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right) \text{ s.t. } x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + \frac{1}{2} \cot u = 0$$

Given,

$$u = \cos^{-1} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$$

$$\cos U = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$$

$$\text{let, } Z = \cos U = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$\therefore Z$ is a homogenous function in terms of x and y
of degree $\frac{1}{2}$

$$x \cdot \frac{\partial Z}{\partial x} + y \cdot \frac{\partial Z}{\partial y} = nZ$$

$$x \cdot \frac{\partial}{\partial x} (\cos U) + y \cdot \frac{\partial}{\partial y} (\cos U) = \frac{1}{2} (\cos U)$$

$$x \cdot (-\sin U) \cdot \frac{\partial U}{\partial x} + y \cdot (-\sin U) \cdot \frac{\partial U}{\partial y} = \frac{1}{2} (\cos U)$$

$$-\sin U \left[x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} \right] = \frac{1}{2} (\cos U)$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = -\frac{1}{2} \frac{\cos U}{\sin U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = -\frac{1}{2} \cot U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + \frac{1}{2} \cot U = 0$$

Hence proved //

5. If $U = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ ST $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{1}{4} \sin 2U$

Given, $U = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\tan U = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

let $z = \tan U = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

$\therefore z$ is a homogenous function of degree $\frac{1}{2}$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n z$$

$$x \cdot \frac{\partial}{\partial x} (\tan U) + y \cdot \frac{\partial}{\partial y} (\tan U) = n (\tan U)$$

$$x (\sec^2 U) \cdot \frac{\partial U}{\partial x} + y (\sec^2 U) \cdot \frac{\partial U}{\partial y} = n \cdot \tan U$$

$$\sec^2 U \left[x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} \right] = n \cdot \tan U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{1}{2} \cdot \frac{\tan U}{\sec^2 U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{1}{2} \cdot \tan U \cdot \sec^{-2} U$$

$$\sin U \quad \sin^2 U \times$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{2} \tan u \cdot \frac{1}{\sec^2 u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{\sin u}{\cos u} \cdot \cos^2 u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{2} \sin u \cdot \cos u$$

multiply & Divide with 2

$$= \frac{1}{2} \frac{2 \sin u \cdot \cos u}{2}$$

$$= \frac{1}{4} \sin 2u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{4} \sin 2u$$

Hence proved //

⑥ If $u = \sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$ ST. $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 0$

Given, $u = \sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$

$$u(x, y) = \sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$$

$$= \sin^{-1} \left(\frac{\sqrt{x} \sqrt{x} - \sqrt{y} \sqrt{y}}{\sqrt{x} \sqrt{x} + \sqrt{y} \sqrt{y}} \right)$$

$$= \sin^{-1} \left(\frac{\sqrt{x}(\sqrt{x}-\sqrt{y})}{\sqrt{x}(\sqrt{x}+\sqrt{y})} \right)$$

$$v(x, y) = \sin^{-1} \left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \right)$$

$$v(x, y) = v$$

v is a homogenous function of degree 0

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = n v$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = 0$$

hence proved //

Imp ** 99%

2019 Example

If $U = \tan^{-1} \left(\frac{x^3+y^3}{x+y} \right)$ Then show that

$$(i) \quad x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \sin 2U$$

$$(ii) \quad x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} = (1 - 4\sin^2 U) \sin 2U$$

Given,

$$U = \tan^{-1} \left(\frac{x^3+y^3}{x+y} \right)$$

$$\tan U = \frac{x^3+y^3}{x+y}$$

$$\text{let } z = \tan U = \frac{x^3 + y^3}{x + y}$$

$\therefore z$ is a homogenous function in terms of x and y of degree 2.

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

$$x \cdot \frac{\partial}{\partial x}(\tan U) + y \cdot \frac{\partial}{\partial y}(\tan U) = 2 \tan U$$

$$x \cdot \sec^2 U \cdot \frac{\partial U}{\partial x} + y \cdot \sec^2 U \cdot \frac{\partial U}{\partial y} = 2 \cdot \tan U$$

$$\sec^2 U \left[x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} \right] = 2 \tan U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2 \cdot \tan U \cdot \frac{1}{\sec^2 U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2 \cdot \frac{\sin U}{\cos U} \cdot \cos^2 U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2 \cdot \sin U \cdot \cos U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \sin 2U \quad \text{--- (1)}$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = \sin 2v$$

p.d wrto x.

$$x \cdot \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \cdot \frac{\partial}{\partial x} (x) + y \cdot \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = \frac{\partial}{\partial x} (\sin 2v)$$

$$x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} (1) + y \cdot \frac{\partial^2 v}{\partial x \partial y} = \cos 2v \cdot \frac{\partial}{\partial x} (2v)$$

$$x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y \cdot \frac{\partial^2 v}{\partial x \partial y} = \cos 2v \cdot 2 \cdot \frac{\partial v}{\partial x}$$

$$x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y \cdot \frac{\partial^2 v}{\partial x \partial y} = 2 \cdot \cos 2v \cdot \frac{\partial v}{\partial x}$$

Multiple both sides with x

$$x^2 \cdot \frac{\partial^2 v}{\partial x^2} + x \cdot \frac{\partial v}{\partial x} + xy \cdot \frac{\partial^2 v}{\partial x \partial y} = 2x \cdot \cos 2v \cdot \frac{\partial v}{\partial x} \quad \text{--- (2)}$$

Similarly,

$$y^2 \cdot \frac{\partial^2 v}{\partial y^2} + y \cdot \frac{\partial v}{\partial y} + xy \cdot \frac{\partial^2 v}{\partial x \partial y} = 2y \cdot \cos 2v \cdot \frac{\partial v}{\partial y} \quad \text{--- (3)}$$

Adding eq (2) & (3)

$$\begin{aligned}
 x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} + xy \frac{\partial^2 u}{\partial x \partial y} \\
 = 2x \cos 2u \cdot \frac{\partial u}{\partial x} + 2y \cos 2u \cdot \frac{\partial u}{\partial y}
 \end{aligned}$$

$$\begin{aligned}
 x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \\
 = 2 \cos 2u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right]
 \end{aligned}$$

from eq ①

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \sin 2u = 2 \cos 2u [\sin 2u]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \cos 2u [\sin 2u] - \sin 2u$$

$$= \sin 2u [-1 + 2 \cos 2u]$$

$$= \sin 2u [2 \cos 2u - 1]$$

$$= \sin 2u [2[1 - 2 \sin^2 u] - 1]$$

$$= \sin 2u [2 - 4 \sin^2 u - 1]$$

$$= \sin 2u [1 - 4 \sin^2 u]$$

$$\boxed{\cos 2\theta = 1 - 2 \sin^2 \theta}$$

$$x \cdot \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 \tilde{u}}{\partial y^2} = \sin 2U [1 - 4 \sin^2 U]$$

Hence proved //

Let $U = \sin^{-1} \left[\frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}} \right]^{1/2}$ Then show that

$$x \cdot \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 \tilde{u}}{\partial y^2} = \frac{\tan U}{144} [13 + \tan^2 U]$$

Given, $U = \sin^{-1} \left[\frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}} \right]^{1/2}$

$$\sin U = \left[\frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}} \right]^{1/2}$$

Let, $Z = \sin U = \left[\frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}} \right]^{1/2}$

$$\begin{aligned} & \left[\frac{x^{1/3}}{x^{1/2}} \right]^{1/2} \\ & \quad \frac{1/3}{1/2} = \frac{2}{3} \\ & \left[\frac{x^{2/3}}{x^{1/2}} \right]^{1/2} \\ & \quad \left[x^{2/3 - 1/2} \right]^{1/2} \\ & \quad \left[x^{1/6} \right]^{1/2} \\ & \quad \Rightarrow x^{1/12} \end{aligned}$$

Z is a homogenous function of degree $-\frac{1}{12}$

By the Euler's theorem

$$x \cdot \frac{\partial Z}{\partial x} + y \cdot \frac{\partial Z}{\partial y} = n Z$$

i.e., $x \cdot \frac{\partial Z}{\partial x} + y \cdot \frac{\partial Z}{\partial y} = n Z$

$$x \cdot \frac{\partial}{\partial x}(\sin u) + y \cdot \frac{\partial}{\partial y}(\sin u) = \frac{-1}{12} \sin u$$

$$x \cdot (\cos u) \cdot \frac{\partial u}{\partial x} + y \cdot \cos u \cdot \frac{\partial u}{\partial y} = \frac{-1}{12} \sin u$$

$$\cos u \left[x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} \right] = \frac{-1}{12} \sin u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{-1}{12} \frac{\sin u}{\cos u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{-1}{12} \tan u \quad \text{--- (1)}$$

p.d wrto x

$$x \cdot \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial x}(x) + y \cdot \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{-1}{12} \cdot \frac{\partial}{\partial x}(\tan u)$$

$$x \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}(1) + y \cdot \frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12} (\sec^2 u) \cdot \frac{\partial u}{\partial x}$$

multiple with x on b.s

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + x \cdot \frac{\partial u}{\partial x} + xy \cdot \frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12} x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x}$$

--- (2)

similarly,

$$y^2 \frac{\partial^2 u}{\partial y^2} + y \cdot \frac{\partial u}{\partial y} + xy \cdot \frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12} y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} \quad \text{--- (3)}$$

Adding eq (2) & (3)

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + x \cdot \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + y \cdot \frac{\partial u}{\partial y} + xy \frac{\partial^2 u}{\partial x \partial y} \\ = \frac{-1}{12} x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} - \frac{1}{12} y \sec^2 u \frac{\partial u}{\partial y} \end{aligned}$$

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \left(x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} \right) \\ = \frac{-1}{12} \sec^2 u \left[x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} \right] \end{aligned}$$

from eq (1)

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \left(\frac{-1}{12} \tan u \right) \\ = \frac{-1}{12} \sec^2 u \left[\frac{-1}{12} \tan u \right] \end{aligned}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{144} \sec^2 u \cdot \tan u + \frac{1}{12} \tan u$$

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} = \frac{1}{12} \tan U \left[\frac{1}{12} \sec^2 U + 1 \right]$$

$$= \frac{1}{12} \tan U \left[\frac{\sec^2 U + 12}{12} \right]$$

$$= \frac{1}{12} \tan U \left[\frac{1 + \tan^2 U + 12}{12} \right]$$

$$\boxed{\sec^2 \theta = 1 + \tan^2 \theta}$$

$$= \frac{1}{144} \tan U \left[\tan^2 U + 13 \right]$$

$$= \frac{\tan U}{144} \left[13 + \tan^2 U \right]$$

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} = \frac{\tan U}{144} \left[13 + \tan^2 U \right]$$

209 (4m)

Hence proved //

If $U = \tan^{-1}\left(\frac{y^2}{x}\right)$. Then find the value of

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2}$$

Given, $U = \tan^{-1}\left(\frac{y^2}{x}\right)$

$$\tan U = \frac{y^2}{x}$$

let, $z = \tan U = \frac{y^2}{x^2}$

$$z = \tan U = \frac{y^2}{x}$$

z is a homogenous function of degree 1

By Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

$$x \cdot \frac{\partial}{\partial x} (\tan U) + y \cdot \frac{\partial}{\partial y} (\tan U) = 1 \cdot (\tan U)$$

$$x \cdot \sec^2 U \cdot \frac{\partial U}{\partial x} + y \cdot \sec^2 U \cdot \frac{\partial U}{\partial y} = \tan U$$

$$\sec^2 U \left[x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} \right] = \frac{\tan U}{\sec^2 U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{\tan U}{\sec^2 U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{\sin U}{\cos U} \cdot \cos^2 U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \sin U \cdot \cos U \quad \text{--- ①}$$

P.D w.r.to x

$$x \cdot \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \cdot \frac{\partial}{\partial x} (x) + y \cdot \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = \frac{\partial}{\partial x} (\sin u \cdot \cos u)$$

$\begin{matrix} u & v \end{matrix}$

$$x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} (1) + y \cdot \frac{\partial^2 v}{\partial x \partial y} = \sin u \cdot \frac{\partial}{\partial x} (\cos u) + \cos u \cdot \frac{\partial}{\partial x} (\sin u)$$

$$x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y \cdot \frac{\partial^2 v}{\partial x \partial y} = \sin u (-\sin u) \cdot \frac{\partial v}{\partial x} + \cos u (\cos u) \cdot \frac{\partial v}{\partial x}$$

$$x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y \cdot \frac{\partial^2 v}{\partial x \partial y} = -\sin^2 u \cdot \frac{\partial v}{\partial x} + \cos^2 u \cdot \frac{\partial v}{\partial x}$$

$$x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y \cdot \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial v}{\partial x} [\cos^2 u - \sin^2 u]$$

multiple x on b.s

$$x^2 \cdot \frac{\partial^2 v}{\partial x^2} + x \cdot \frac{\partial v}{\partial x} + xy \cdot \frac{\partial^2 v}{\partial x \partial y} = x \cdot \frac{\partial v}{\partial x} [\cos^2 u - \sin^2 u]$$

— (2)

similarly,

$$y^2 \cdot \frac{\partial^2 v}{\partial y^2} + y \cdot \frac{\partial v}{\partial y} + xy \cdot \frac{\partial^2 v}{\partial x \partial y} = y \cdot \frac{\partial v}{\partial y} [\cos^2 u - \sin^2 u]$$

— (3)

Adding eq (2) & (3)

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + x \cdot \frac{\partial \tilde{u}}{\partial x} + xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 \tilde{u}}{\partial x^2} + y \cdot \frac{\partial \tilde{u}}{\partial x} + xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y}$$

$$= x \cdot \frac{\partial \tilde{u}}{\partial x} [\cos^2 \tilde{u} - \sin^2 \tilde{u}] + y \cdot \frac{\partial \tilde{u}}{\partial y} [\cos^2 \tilde{u} - \sin^2 \tilde{u}]$$

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 \tilde{u}}{\partial x^2} + \left[x \cdot \frac{\partial \tilde{u}}{\partial x} + y \cdot \frac{\partial \tilde{u}}{\partial y} \right]$$

$$= \cos^2 \tilde{u} - \sin^2 \tilde{u} \left[x \cdot \frac{\partial \tilde{u}}{\partial x} + y \cdot \frac{\partial \tilde{u}}{\partial y} \right]$$

from eq ①

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 \tilde{u}}{\partial x^2} + [\sin \tilde{u} \cdot \cos \tilde{u}]$$

$$= \cos^2 \tilde{u} - \sin^2 \tilde{u} [\sin \tilde{u} \cdot \cos \tilde{u}]$$

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 \tilde{u}}{\partial x^2} = \cos^2 \tilde{u} - \sin^2 \tilde{u} [\sin \tilde{u} \cdot \cos \tilde{u}] - [\sin \tilde{u} \cdot \cos \tilde{u}]$$

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 \tilde{u}}{\partial x^2} = \sin \tilde{u} \cdot \cos \tilde{u} [\cos^2 \tilde{u} - \sin^2 \tilde{u}]$$

map with 2

$$= \frac{1}{2} 2 \sin \tilde{u} \cos \tilde{u} \underbrace{[1 - \sin^2 \tilde{u} - \sin^2 \tilde{u}]}_{1 - 2 \sin^2 \tilde{u}}$$

$$= \frac{1}{2} \sin 2\tilde{u} [\cos 2\tilde{u} - 1]$$

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 \tilde{u}}{\partial x^2} = \frac{1}{2} \sin 2\tilde{u} [\cos 2\tilde{u} - 1]$$

$\sin 2\theta = 2 \sin \theta \cos \theta$ $\cos 2\theta = 1 - 2 \sin^2 \theta$
--

④ If $z = (x+y) \phi(y/x)$ where ϕ is a arbitrary function

Prove that $x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z$

Given, $z = (x+y) \phi(y/x)$

$$z(kx, ky) = (kx + ky) \phi\left(\frac{ky}{kx}\right)$$

$$= k(x+y) \phi\left(\frac{y}{x}\right)$$

$$= k z$$

$\therefore z$ is a homogenous function of degree 1

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 1 \cdot z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z$$

Hence proved //

24 If $U = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ then show that

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 0$$

Given, $U = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$U(kx, ky) = \sin^{-1}\left(\frac{kx}{ky}\right) + \tan^{-1}\left(\frac{ky}{kx}\right)$$

$$= \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$U(kx, ky) = U$$

U is a homogenous function of degree 0.

By Euler's theorem

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = n U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 0 U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 0 \cdot U$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 0$$

Hence proved //

NOTE:

$$\text{If } v = \log a \Rightarrow a = e^v$$

Example: If $v = \log\left(\frac{x^4 + y^4}{x + y}\right)$ then show that

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = 3$$

Given,

$$v = \log\left(\frac{x^4 + y^4}{x + y}\right)$$

$$e^v = \frac{x^4 + y^4}{x + y}$$

$$\text{Let } z = e^v = \frac{x^4 + y^4}{x + y}$$

z is a homogenous function of degree 3

By the Euler's Theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

$$x \cdot \frac{\partial}{\partial x}(e^v) + y \cdot \frac{\partial}{\partial y}(e^v) = 3e^v$$

$$x \cdot e^v \cdot \frac{\partial v}{\partial x} + y \cdot e^v \cdot \frac{\partial v}{\partial y} = 3e^v$$

$$e^v \left[x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} \right] = 3e^v$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = \frac{3e^v}{e^v}$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = 3$$

Hence proved //

If $v = \log \left[\frac{x^2 + y^2}{x + y} \right]$. Then show that

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = 1$$

Given, $v = \log \left[\frac{x^2 + y^2}{x + y} \right]$

$$e^v = \frac{x^2 + y^2}{x + y}$$

Let, $z = e^v = \frac{x^2 + y^2}{x + y}$

z is a homogenous function of degree 1

By the Euler's Theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

$$x \cdot \frac{\partial}{\partial x}(e^v) + y \cdot \frac{\partial}{\partial y}(e^v) = 1 \cdot e^v$$

$$x \cdot e^v \cdot \frac{\partial v}{\partial x} + y \cdot e^v \cdot \frac{\partial v}{\partial y} = e^v$$

$$e^u \left[x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} \right] = e^u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{e^u}{e^u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 1$$

Hence proved //

⊙ If $z = \sec^{-1} \left(\frac{x^3 + y^3}{x + y} \right)$ s.t. $x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2z$

Given, $z = \sec^{-1} \left(\frac{x^3 + y^3}{x + y} \right)$

let $u = z = \sec^{-1} \left(\frac{x^3 + y^3}{x + y} \right)$

$$u = \sec z = \frac{x^3 + y^3}{x + y}$$

u is a homogenous function of degree 2

By the Euler's theorem

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu$$

$$x \cdot \frac{\partial}{\partial x} (\sec z) + y \cdot \frac{\partial}{\partial y} (\sec z) = 2 \sec z$$

$$x \cdot \sec z \cdot \tan z \cdot \frac{\partial z}{\partial x} + y \cdot \sec z \cdot \tan z \cdot \frac{\partial z}{\partial y} = 2 \sec z$$

$$\sec z \tan z \left(x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right) = 2 \sec z$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2 \sec z \cdot \frac{1}{\sec z \tan z}$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2 \cdot \frac{1}{\tan z}$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2 \cot z$$

Hence proved //

If $\sin v = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$ Then S.T

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + z \cdot \frac{\partial v}{\partial z} = -3 \tan v$$

Given that,

$$\sin v = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$$

$$\text{let } u = \sin v = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$$

$$U(kx, ky, kz) = \frac{kx + 2ky + 3kz}{\sqrt{k^8 x^8 + k^8 y^8 + k^8 z^8}}$$

$$= \frac{k(x + 2y + 3z)}{\sqrt{k^8} \cdot \sqrt{x^8 + y^8 + z^8}}$$

$$= \frac{k}{k^4} U(x, y, z)$$

$$U(kx, ky, kz) = k^{-3} U(x, y, z)$$

$\therefore U$ is a homogenous function of degree -3

By the Euler's theorem

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = nU$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = nU$$

$$x \cdot \frac{\partial}{\partial x}(\sin v) + y \cdot \frac{\partial}{\partial y}(\sin v) + z \cdot \frac{\partial}{\partial z}(\sin v) = -3(\sin v)$$

$$x \cdot \cos v \cdot \frac{\partial v}{\partial x} + y \cdot \cos v \cdot \frac{\partial v}{\partial y} + z \cdot \cos v \cdot \frac{\partial v}{\partial z} = -3(\sin v)$$

$$\cos v \left[x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + z \cdot \frac{\partial v}{\partial z} \right] = -3 \sin v$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + z \cdot \frac{\partial v}{\partial z} = -3 \frac{\sin v}{\cos v}$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + z \cdot \frac{\partial v}{\partial z} = -3 \tan v$$

Hence proved

Ex 10. Put $U = \tan^{-1}\left(\frac{y^2}{x}\right)$. Then find the value of

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2}$$

Given,

$$U = \tan^{-1}\left(\frac{y^2}{x}\right)$$

$$\tan U = \frac{y^2}{x}$$

$$\text{Let } z = \tan U = \frac{y^2}{x}$$

z is a homogenous function of degree 1

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n z$$

$$x \cdot \frac{\partial}{\partial x} (\tan U) + y \cdot \frac{\partial}{\partial y} (\tan U) = 1 \cdot \tan U$$

$$x \cdot \sec^2 U \cdot \frac{\partial U}{\partial x} + y \cdot \sec^2 U \cdot \frac{\partial U}{\partial y} = \tan U$$

$$\sec^2 U \left[x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} \right] = \tan U$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = \tan u \cdot \frac{1}{\sec^2 u}$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = \tan u \cdot \frac{\sin u}{\cos u} \cdot \cos^2 u$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = \sin u \cdot \cos u$$

made with 2

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = \frac{1}{2} 2 \sin u \cos u$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = \frac{1}{2} \sin 2u \quad \text{--- (1)}$$

p.d. w.r.to x

$$x \cdot \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \cdot \frac{\partial}{\partial x} (x) + y \cdot \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = \frac{1}{2} \cdot \frac{\partial}{\partial x} (\sin 2u)$$

$$x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} (1) + y \cdot \frac{\partial^2 v}{\partial x \partial y} = \frac{1}{2} \cos 2u \cdot 2 \cdot \frac{\partial v}{\partial x}$$

$$x^2 \cdot \frac{\partial^2 v}{\partial x^2} + x \cdot \frac{\partial v}{\partial x} + xy \cdot \frac{\partial^2 v}{\partial x \partial y} = \frac{1}{2} x \cdot \cos 2u \cdot \frac{\partial v}{\partial x} \quad \text{--- (2)}$$

similarly,

$$y \cdot \frac{\partial^2 v}{\partial y^2} + y \cdot \frac{\partial v}{\partial y} + xy \cdot \frac{\partial^2 v}{\partial x \partial y} = y \cdot \cos 2u \cdot \frac{\partial v}{\partial y} \quad \text{--- (3)}$$

Adding eq (2) & (3)

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + x \cdot \frac{\partial \tilde{u}}{\partial x} + xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \frac{\partial^2 \tilde{u}}{\partial y^2} + y \cdot \frac{\partial \tilde{u}}{\partial y} + xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y}$$

$$= x \cos 2u \cdot \frac{\partial \tilde{u}}{\partial x} + y \cos 2u \cdot \frac{\partial \tilde{u}}{\partial y}$$

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \frac{\partial^2 \tilde{u}}{\partial y^2} + \left(x \cdot \frac{\partial \tilde{u}}{\partial x} + y \cdot \frac{\partial \tilde{u}}{\partial y} \right)$$

$$= \cos 2u \left[x \cdot \frac{\partial \tilde{u}}{\partial x} + y \cdot \frac{\partial \tilde{u}}{\partial y} \right]$$

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \frac{\partial^2 \tilde{u}}{\partial y^2} + \left[\frac{1}{2} \sin 2u \right]$$

$$= \cos 2u \left[\frac{1}{2} \sin 2u \right]$$

from eq (1)

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \frac{\partial^2 \tilde{u}}{\partial y^2} = \cos 2u \left[\frac{1}{2} \sin 2u \right] - \frac{1}{2} \sin 2u$$

$$= \frac{1}{2} \sin 2u [\cos 2u - 1]$$

$$x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + 2xy \cdot \frac{\partial^2 \tilde{u}}{\partial x \partial y} + y^2 \frac{\partial^2 \tilde{u}}{\partial y^2} = \frac{1}{2} \sin 2u [\cos 2u - 1]$$

Example

① If $U = z \cdot e^{ax+by}$ where z is a homogenous function in x and y of degree n . Then s.t

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = (ax + by + n) U.$$

Given,

$$U = z \cdot e^{ax+by}$$

Given that z is a homogenous function in x & y of degree n

By the Euler's theorem

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz \quad \text{--- ①}$$

$$\text{LHS} = x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y}$$

$$= x \cdot \frac{\partial}{\partial x} [z \cdot e^{ax+by}] + y \cdot \frac{\partial}{\partial y} [z \cdot e^{ax+by}]$$

$$= x \left\{ z \cdot \frac{\partial}{\partial x} (e^{ax+by}) + e^{ax+by} \frac{\partial z}{\partial x} \right\} + y \left\{ z \cdot \frac{\partial}{\partial y} (e^{ax+by}) + e^{ax+by} \frac{\partial z}{\partial y} \right\}$$

$$= x \left\{ z \cdot e^{ax+by} \cdot \frac{\partial}{\partial x} (ax+by) + e^{ax+by} \frac{\partial z}{\partial x} \right\} + y \left\{ z \cdot e^{ax+by} \cdot \frac{\partial}{\partial y} (ax+by) + e^{ax+by} \frac{\partial z}{\partial y} \right\}$$

$$x \left[z \cdot e^{ax+by} \cdot a + e^{ax+by} \cdot \frac{\partial z}{\partial x} \right] + y \left[z \cdot e^{ax+by} \cdot b + e^{ax+by} \cdot \frac{\partial z}{\partial y} \right]$$

$$x \cdot z \cdot e^{ax+by} \cdot a + x e^{ax+by} \cdot \frac{\partial z}{\partial x} + y z \cdot e^{ax+by} \cdot b + y e^{ax+by} \cdot \frac{\partial z}{\partial y}$$

$$e^{ax+by} \left[xza + x \cdot \frac{\partial z}{\partial x} + yz \cdot b + y \cdot \frac{\partial z}{\partial y} \right]$$

$$e^{ax+by} \left[x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} + xza + yzb \right]$$

$$= e^{ax+by} [nz + xza + yzb] \quad [\text{from eq 1}]$$

$$= e^{ax+by} z [n + ax + by]$$

$$= u [ax + by + n]$$

$$= [ax + by + n] u$$

$$= \text{RHS}$$

Hence proved. //

Example:

$$\text{If } v = \log \sin \left[\frac{\pi (2x^2 + y^2 + xz)^{1/2}}{2(x^2 + xy + 2yz + z^2)^{1/3}} \right] \quad \text{Find the value}$$

$$q \quad x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + z \cdot \frac{\partial v}{\partial z} \quad \text{at } x=0 \quad y=1 \quad z=2$$

$$\text{Given, } v = \log \sin \left[\frac{\pi (2x^2 + y^2 + xz)^{1/2}}{2(x^2 + xy + 2yz + z^2)^{1/3}} \right]$$

let,

$$U = \frac{\pi(2x^2 + y^2 + xz)^{1/2}}{2(x^2 + xy + 2yz + z^2)^{1/3}}$$

$$\frac{(k^2)^{1/2}}{(k^2)^{1/3}}$$

$$= \frac{k}{k^{1/3}}$$

$$= k^{1-1/3}$$

$$= k^{2/3}$$

$\therefore U$ is a homogenous function of degree $\frac{1}{3}$

By the Euler's theorem

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + z \cdot \frac{\partial U}{\partial z} = \frac{1}{3} U \quad \text{--- (1)}$$

$$U = \frac{\pi(1)}{2(8)^{1/3}}$$

$$= \frac{\pi}{2(2^3)^{1/3}}$$

$$= \frac{\pi}{2 \times 2}$$

$$U = \frac{\pi}{4}$$

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} + z \cdot \frac{\partial v}{\partial z}$$

$$\Rightarrow x \cdot \frac{\partial}{\partial x} [\log(\sin U)] + y \cdot \frac{\partial}{\partial y} [\log(\sin U)] + z \cdot \frac{\partial}{\partial z} [\log(\sin U)]$$

$$\Rightarrow x \cdot \frac{1}{\sin U} \cdot \frac{\partial}{\partial x} (\sin U) + y \cdot \frac{1}{\sin U} \cdot \frac{\partial}{\partial y} (\sin U) + z \cdot \frac{1}{\sin U} \cdot \frac{\partial}{\partial z} (\sin U)$$

$$= x \cdot \frac{1}{\sin u} \cos u \cdot \frac{\partial u}{\partial x} + y \cdot \frac{1}{\sin u} \cos u \cdot \frac{\partial u}{\partial y} + z \cdot \frac{1}{\sin u} \cos u \cdot \frac{\partial u}{\partial z}$$

$$= x \cdot \cot u \cdot \frac{\partial u}{\partial x} + y \cdot \cot u \cdot \frac{\partial u}{\partial y} + z \cdot \cot u \cdot \frac{\partial u}{\partial z}$$

$$= \cot u \left[x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} \right]$$

$$= \cot u \left[\frac{1}{3} u \right] \text{ from eq ①}$$

$$= \cot \frac{\pi}{4} \cdot \frac{1}{3} \cdot \frac{\pi}{4} \quad \text{at } x=0, y=1, z=2$$

$$= 1 \times \frac{\pi}{12}$$

$$\therefore x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = \frac{\pi}{12}$$

If $U = \frac{x^2 y^2}{x^2 + y^2}$ show that

$$(1) x \cdot \frac{\partial^2 U}{\partial x^2} + y \cdot \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial U}{\partial x}$$

$$(2) x \cdot \frac{\partial^2 U}{\partial x \partial y} + y \cdot \frac{\partial^2 U}{\partial y^2} = \frac{\partial U}{\partial y}$$

$$(3) x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} = 2U$$

Given that,

$$U = \frac{x^2 y^2}{x^2 + y^2}$$

$$U(x, y) = \frac{k^2 x^2 \cdot k y^2}{k^2 x^2 + k y^2}$$

$$= \frac{k^4 x^2 y^2}{k^2 [x^2 + y^2]}$$

$$= k^2 \cdot \frac{x^2 y^2}{x^2 + y^2}$$

$$= k^2 \cdot U$$

$\therefore U$ is a homogenous function of degree 2.

$$(i) \quad \boxed{x \cdot \frac{\partial^2 U}{\partial x^2} + y \cdot \frac{\partial^2 U}{\partial x \partial y} = (n-1) \frac{\partial U}{\partial x}}$$

$$x \cdot \frac{\partial^2 U}{\partial x^2} + y \cdot \frac{\partial^2 U}{\partial x \partial y} = (2-1) \cdot \frac{\partial U}{\partial x}$$

$$x \cdot \frac{\partial^2 U}{\partial x^2} + y \cdot \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial U}{\partial x}$$

$$(ii) \quad \boxed{y \cdot \frac{\partial^2 U}{\partial y^2} + x \cdot \frac{\partial^2 U}{\partial x \partial y} = (n-1) \frac{\partial U}{\partial y}}$$

$$y \cdot \frac{\partial^2 U}{\partial y^2} + x \cdot \frac{\partial^2 U}{\partial x \partial y} = (2-1) \cdot \frac{\partial U}{\partial y}$$

$$y \cdot \frac{\partial^2 U}{\partial y^2} + x \cdot \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial U}{\partial y}$$

$$\begin{aligned}
 x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= n(n-1)u \\
 &= 2(2-1)u \\
 &= 2 \times 1 \times u \\
 &= 2u
 \end{aligned}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u.$$

Hence proved //

Example:

If $z = x^m f\left(\frac{y}{x}\right) + x^n g\left(\frac{x}{y}\right)$ prove that

$$\begin{aligned}
 x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz \\
 = [m+n+1] \left[x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right]
 \end{aligned}$$

$$z = x^m f\left(\frac{y}{x}\right) + x^n g\left(\frac{x}{y}\right) \quad \text{--- (1)}$$

let $u = x^m f\left(\frac{y}{x}\right)$

u is a homogenous function
of degree m .

By the Euler's Theorem

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = mu. \quad \text{--- (2)}$$

$$v = x^n g\left(\frac{x}{y}\right)$$

v is a homogenous
function of degree n

By the Euler's Theorem

$$x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = n \cdot v. \quad \text{--- (3)}$$

from eq (1)

$$Z = U + V$$

Adding eq (2) & (3)

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} + x \cdot \frac{\partial V}{\partial x} + y \cdot \frac{\partial V}{\partial y} = mU + nV$$

$$x \left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial x} \right] + y \left[\frac{\partial U}{\partial y} + \frac{\partial V}{\partial y} \right] = mU + nV$$

$$x \cdot \frac{\partial [U+V]}{\partial x} + y \cdot \frac{\partial [U+V]}{\partial y} = mU + nV$$

$$x \cdot \frac{\partial Z}{\partial x} + y \cdot \frac{\partial Z}{\partial y} = mU + nV \quad \text{--- (4)}$$

$$U = x^m \cdot f\left(\frac{y}{x}\right)$$

$$V = x^n \cdot g\left(\frac{x}{y}\right)$$

By second order Euler's Theorem

$$x^2 \cdot \frac{\partial^2 U}{\partial x^2} + 2xy \cdot \frac{\partial^2 U}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 U}{\partial y^2} = m(m-1)U \quad \text{--- (5)}$$

and

$$x^2 \cdot \frac{\partial^2 V}{\partial x^2} + 2xy \cdot \frac{\partial^2 V}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 V}{\partial y^2} = n(n-1)V \quad \text{--- (6)}$$

Adding eq (5) & (6)

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2}$$

$$= m(m-1)u + n(n-1)v$$

$$x^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \right] + 2xy \left[\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} \right] + y^2 \left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \right]$$

$$= (m^2 - m)u + (n^2 - n)v$$

$$x^2 \frac{\partial^2 (u+v)}{\partial x^2} + 2xy \frac{\partial^2 (u+v)}{\partial x \partial y} + y^2 \frac{\partial^2 (u+v)}{\partial y^2}$$

$$= m^2 u - mu + n^2 v - nv$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \underbrace{[m^2 u + n^2 v]}_{(mu+nv)} - (mu+nv)$$

$$= m(m+n)u + n(m+n)v - mn(u+v) - (mu+nv)$$

$$= (m+n)[mu+nv] - mn(z) - (mu+nv)$$

$$= (mu+nv)[m+n-1] - mnz$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = \left[x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right]$$

from (4) $(m+n-1)$

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = \left[x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right] [m+n-1]$$

Hence proved.

2020 Essay

If $U = f(r)$ where $r = \sqrt{x^2 + y^2}$ then show that

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

GT $U = f(r)$

P.D. w.r.to x

$$\frac{\partial U}{\partial x} = f'(r) \frac{\partial}{\partial x}(r)$$

$$= f'(r) \cdot \frac{\partial}{\partial x} (\sqrt{x^2 + y^2})$$

$$= f'(r) \frac{\partial}{\partial x} (x^2 + y^2)^{\frac{1}{2}}$$

$$= f'(r) \cdot \frac{1}{2} (x^2 + y^2)^{\frac{1}{2} - 1} \cdot \frac{\partial}{\partial x} (x^2 + y^2)$$

$$= f'(r) \cdot \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \cdot (2x)$$

$$\frac{\partial U}{\partial x} = f'(r) \cdot (x^2 + y^2)^{-\frac{1}{2}} (x)$$

$$\left[\frac{\partial}{\partial x} (uvw) = uw' + uv' + uvw' \right]$$

P.D. w.r.to x

$$\frac{\partial^2 U}{\partial x^2} = f'(r) (x^2 + y^2)^{-\frac{1}{2}} \frac{\partial}{\partial x} (x) + f'(r) \cdot x \cdot \frac{\partial}{\partial x} (x^2 + y^2)^{-\frac{1}{2}} +$$

$$(x^2 + y^2)^{-\frac{1}{2}} x \cdot \frac{\partial}{\partial x} (f'(r))$$

$$= f'(r) (x^2 + y^2)^{-\frac{1}{2}} (1) + f'(r) \cdot x \cdot \frac{-1}{2} (x^2 + y^2)^{-\frac{1}{2} - 1} \cdot \frac{\partial}{\partial x} (x^2 + y^2) +$$

$$(x^2 + y^2)^{-\frac{1}{2}} x \cdot f''(r) \cdot \frac{\partial}{\partial x} (r)$$

$$f'(x)(x^2+y^2)^{-1/2} + f'(x)x \cdot \left(\frac{-1}{2}\right)(x^2+y^2)^{-3/2} (2x) + (x^2+y^2)^{-1/2} \cdot x \cdot f''(x)$$

$$\frac{\partial}{\partial x} (x^2+y^2)^{1/2}$$

$$= f'(x)(x^2+y^2)^{-1/2} - f'(x)x^2(x^2+y^2)^{-3/2} + (x^2+y^2)^{-1/2} x f''(x)$$

$$\frac{1}{2}(x^2+y^2)^{-1/2} (2x)$$

$$\frac{\partial^2 u}{\partial x^2} = f'(x)(x^2+y^2)^{-1/2} - f'(x)x^2(x^2+y^2)^{-3/2} + f''(x)x^2(x^2+y^2)^{-1}$$

— (1)

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = f'(x)(x^2+y^2)^{-1/2} - f'(x)y^2(x^2+y^2)^{-3/2} + f''(x)y^2(x^2+y^2)^{-1}$$

— (2)

Adding eq (1) & (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2f'(x)(x^2+y^2)^{-1/2} - f'(x)(x^2+y^2)^{-3/2} [x^2+y^2] + f''(x)(x^2+y^2)^{-1} [x^2+y^2]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2f'(r)(x^2+y^2)^{-1/2} - f'(r)(x^2+y^2)^{-3/2+1} + f''(r)(x^2+y^2)^{-1/2}$$

$$= 2f'(r)(x^2+y^2)^{-1/2} - f'(r)(x^2+y^2)^{-1/2} + f''(r)(x^2+y^2)^0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f'(r)(x^2+y^2)^{-1/2} [2-1] + f''(r)(1)$$

$$= f'(r)(x^2+y^2)^{-1/2} + f''(r)$$

$$= f'(r) \frac{1}{(x^2+y^2)^{1/2}} + f''(r)$$

$$= f'(r) \frac{1}{\sqrt{x^2+y^2}} + f''(r)$$

$$= f'(r) \cdot \frac{1}{r} + f''(r)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} \cdot f'(r) //$$

Hence proved.

Example:

If $v = A t^{-1/2} e^{-\frac{x^2}{4a^2 t}}$ prove that

$$\frac{\partial v}{\partial t} = a^2 \cdot \frac{\partial^2 v}{\partial x^2}$$

Given that,

$$v = A t^{-1/2} \cdot e^{\frac{-x^2}{4a^2t}} \quad \text{--- (1)}$$

P.D w.r.to t

$$\frac{\partial v}{\partial t} = A \left[t^{-1/2} \cdot \frac{\partial}{\partial t} \left(e^{\frac{-x^2}{4a^2t}} \right) + e^{\frac{-x^2}{4a^2t}} \cdot \frac{\partial}{\partial t} (t^{-1/2}) \right]$$

$$= A \left[t^{-1/2} \cdot e^{\frac{-x^2}{4a^2t}} \cdot \frac{\partial}{\partial t} \left(\frac{-x^2}{4a^2t} \right) + e^{\frac{-x^2}{4a^2t}} \left(-\frac{1}{2} \right) t^{-3/2} \right]$$

$$= A \left[t^{-1/2} \cdot e^{\frac{-x^2}{4a^2t}} \cdot \left(\frac{-x^2}{4a^2} \right) \frac{\partial}{\partial t} \left(\frac{1}{t} \right) - \frac{1}{2} e^{\frac{-x^2}{4a^2t}} \cdot t^{-3/2} \right]$$

$$= A \left[t^{-1/2} \cdot e^{\frac{-x^2}{4a^2t}} \cdot \left(\frac{-x^2}{4a^2} \right) \left(-\frac{1}{t^2} \right) - \frac{1}{2} e^{\frac{-x^2}{4a^2t}} \cdot t^{-1/2} \cdot t^{-1} \right]$$

$$= A t^{-1/2} \cdot e^{\frac{-x^2}{4a^2t}} \left[\frac{x^2}{4a^2t^2} - \frac{1}{2} t^{-1} \right]$$

from eq (1)

$$= v \left[\frac{x^2}{4a^2t^2} - \frac{1}{2t} \right]$$

$$\frac{\partial v}{\partial t} = v \left[\frac{x^2 - 2a^2t}{4a^2t^2} \right] \quad \text{--- (2)}$$

$$v = A t^{-1/2} \cdot e^{\frac{-x^2}{4a^2t}}$$

P.D w.r.to x

$$\frac{\partial v}{\partial x} = A t^{-1/2} \cdot \frac{\partial}{\partial x} \left(e^{\frac{-x^2}{4a^2t}} \right)$$

$$\frac{\partial v}{\partial x} = A t^{-1/2} e^{\frac{-x^2}{4a^2t}} \cdot \frac{\partial}{\partial x} \left(\frac{-x^2}{4a^2t} \right)$$

$$= A t^{-1/2} e^{\frac{-x^2}{4a^2t}} \left(\frac{-1}{2a^2t} \right) \frac{\partial}{\partial x} (x^2)$$

$$= A t^{-1/2} e^{\frac{-x^2}{4a^2t}} \left(\frac{-1}{2a^2t} \right) (2x)$$

$$\frac{\partial v}{\partial x} = A t^{-1/2} e^{\frac{-x^2}{4a^2t}} \left(\frac{-x}{2a^2t} \right)$$

Again P.D wrto x

$$\frac{\partial^2 v}{\partial x^2} = A t^{-1/2} \left[e^{\frac{-x^2}{4a^2t}} \frac{\partial}{\partial x} \left(\frac{-x}{2a^2t} \right) + \left(\frac{-x}{2a^2t} \right) \cdot \frac{\partial}{\partial x} \left(e^{\frac{-x^2}{4a^2t}} \right) \right]$$

$$= A t^{-1/2} \left[e^{\frac{-x^2}{4a^2t}} \left(\frac{-1}{2a^2t} \right) \frac{\partial}{\partial x} (x) + \left(\frac{-x}{2a^2t} \right) \left(e^{\frac{-x^2}{4a^2t}} \right) \cdot \frac{\partial}{\partial x} \left(\frac{-x^2}{4a^2t} \right) \right]$$

$$= A t^{-1/2} \left[e^{\frac{-x^2}{4a^2t}} \left(\frac{-1}{2a^2t} \right) (1) + \left(\frac{-x}{2a^2t} \right) e^{\frac{-x^2}{4a^2t}} \left(\frac{-1}{2a^2t} \right) \frac{\partial}{\partial x} (x^2) \right]$$

$$= A t^{-1/2} \left[e^{\frac{-x^2}{4a^2t}} \left(\frac{-1}{2a^2t} \right) + \left(\frac{-x}{2a^2t} \right) e^{\frac{-x^2}{4a^2t}} \left(\frac{-1}{2a^2t} \right) (2x) \right]$$

$$\frac{\partial v}{\partial x} = A t^{-1/2} e^{-\frac{x^2}{4a^2 t}} \left(\frac{-x}{2a^2 t} \right)$$

$$\frac{\partial v}{\partial x} = v \left(\frac{-x}{2a^2 t} \right)$$

$$\frac{\partial v}{\partial x} = \frac{-vx}{2a^2 t} \quad \text{--- (3)}$$

Again PD w.r.to x

$$\frac{\partial^2 v}{\partial x^2} = \frac{-1}{2a^2 t} \cdot \frac{\partial}{\partial x} (vx)$$

$$= \frac{-1}{2a^2 t} \left[v \cdot \frac{\partial}{\partial x} (x) + x \cdot \frac{\partial}{\partial x} v \right]$$

$$= \frac{-1}{2a^2 t} \left[v \cdot (1) + x \cdot \frac{\partial v}{\partial x} \right]$$

$$= \frac{-1}{2a^2 t} \left[v + x \left[\frac{-vx}{2a^2 t} \right] \right] \quad \text{from eq (3)}$$

$$= \frac{-v}{2a^2 t} \left[1 + \frac{-x^2}{2a^2 t} \right]$$

$$= \frac{-v}{2a^2 t} \left[1 - \frac{x^2}{2a^2 t} \right]$$

$$= \frac{-v}{2a^4 t} \left[\frac{2a^4 t - x^2}{2a^2 t} \right]$$

$$= \frac{-v}{2a^2t} \left[(-1) \frac{[x^2 - 2a^2t]}{2a^2t} \right]$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{v}{2a^2t} \left[\frac{x^2 - 2a^2t}{2a^2t} \right]$$

multiple a^2 on bs

$$a^2 \cdot \frac{\partial^2 v}{\partial x^2} = a^2 \cdot \frac{v}{2a^2t} \cdot \left[\frac{x^2 - 2a^2t}{2a^2t} \right]$$

$$a^2 \cdot \frac{\partial^2 v}{\partial x^2} = \frac{v}{2t} \left[\frac{x^2 - 2a^2t}{2a^2t} \right]$$

$$a^2 \cdot \frac{\partial^2 v}{\partial x^2} = v \left[\frac{x^2 - 2a^2t}{4a^2t^2} \right] \quad \text{--- (4)}$$

Equating eq (2) & (4)

$$\frac{\partial v}{\partial t} = a^2 \cdot \frac{\partial^2 v}{\partial x^2}$$

Hence proved. //

* If $\theta = t^n \cdot e^{\frac{-r^2}{4t}}$, find the value of n which

will make $\frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left[r^2 \cdot \frac{\partial \theta}{\partial r} \right] = \frac{\partial \theta}{\partial t}$

Given $\theta = t^n e^{-r^2/4t}$

P.D w.r to r

$$\frac{\partial \theta}{\partial r} = t^n \cdot \frac{\partial}{\partial r} e^{-r^2/4t}$$

$$= t^n \cdot e^{-r^2/4t} \cdot \frac{\partial}{\partial r} \left(\frac{-r^2}{4t} \right)$$

$$= t^n \cdot e^{-r^2/4t} \left(\frac{-2r}{4t} \right)$$

$$\frac{\partial \theta}{\partial r} = t^n \cdot e^{-r^2/4t} \left(\frac{-r}{2t} \right)$$

$$r^2 \frac{\partial \theta}{\partial r} = -t^n \cdot \frac{r^3}{2t} e^{-r^2/4t}$$

[Multiply with r^2 on both sides]

P.D w.r to r

$$\frac{\partial}{\partial r} \left[r^2 \frac{\partial \theta}{\partial r} \right] = \frac{-t^{n+1}}{2} \left[r^3 \frac{\partial}{\partial r} e^{-r^2/4t} + e^{-r^2/4t} \cdot \frac{\partial}{\partial r} r^3 \right]$$

$$= \frac{-t^{n+1}}{2} \left[r^3 \cdot e^{-r^2/4t} \left(\frac{-2r}{4t} \right) + e^{-r^2/4t} \cdot (3r^2) \right]$$

$$= \frac{-t^{n+1}}{2} e^{-r^2/4t} \left[r^3 \left(\frac{-2r}{4t} \right) + 3 \right]$$

$$= -\frac{t^{n+1}}{2} r^2 \cdot e^{-r^2/4t} \left[3 - \frac{r^2}{2t} \right]$$

$$\frac{\partial}{\partial r} \left[r^2 \frac{\partial \theta}{\partial r} \right] = -\frac{t^{n+1}}{2} r^2 \cdot e^{-r^2/4t} \left[3 - \frac{r^2}{2t} \right]$$

Divide by r^2

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial \theta}{\partial r} \right] = -\frac{t^{n+1}}{2} e^{-r^2/4t} \left[3 - \frac{r^2}{2t} \right] \quad \text{--- (1)}$$

$$\theta = t^n \cdot e^{-r^2/4t}$$

PD wrt to t

$$\frac{\partial \theta}{\partial t} = t^n \cdot \frac{\partial}{\partial t} \left(e^{-r^2/4t} \right) + e^{-r^2/4t} \frac{\partial}{\partial t} (t^n)$$

$$= t^n \cdot e^{-r^2/4t} \cdot \frac{\partial}{\partial t} \left(\frac{-r^2}{4t} \right) + e^{-r^2/4t} \cdot n t^{n-1}$$

$$= t^n \cdot e^{-r^2/4t} \left[\frac{-r^2}{4} \left(\frac{-1}{t^2} \right) \right] + e^{-r^2/4t} \cdot n t^{n-1}$$

$$= \frac{t^n}{t^2} \cdot \frac{e^{-r^2/4t}}{4} r^2 + e^{-r^2/4t} n t^{n-1}$$

$$\frac{\partial \theta}{\partial t} = t^{n-1} \cdot e^{-r^2/4t} \left[\frac{r^2}{4t} + n \right] \quad \text{--- (2)}$$

Given,

$$\frac{1}{r^2} \left[\frac{\partial}{\partial r} r^2 \frac{\partial \theta}{\partial r} \right] = \frac{\partial \theta}{\partial t}$$

$$\frac{1}{2} \cancel{e^{-r^2/4t}} \left(3 - \frac{r^2}{2t} \right) = \cancel{t^n} \cdot \cancel{e^{-r^2/4t}} \left(\frac{r^2}{4t} + n \right)$$

$$\left(\frac{r^2}{2t} - 3 \right) \frac{1}{2} = \frac{r^2}{4t} + n$$

$$\frac{r^2}{2t} - 3 = 2 \left(\frac{r^2}{4t} + n \right)$$

$$\frac{r^2}{2t} - 3 = \frac{2r^2}{4t} + 2n$$

$$\cancel{\frac{r^2}{2t}} - 3 = \cancel{\frac{r^2}{2t}} + 2n$$

$$2n = -3$$

$$\boxed{n = -3/2}$$