UNIT 2

TOTAL DIFFERENTIATION

Defination of composite function:

Let $z = f(x, y)$ is a function in terms of x and y & $x = \phi(t)$, $y = \psi(t)$ are function in terms of t

Then,
$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

Example 1:

find $\frac{dz}{dt}$ if $z = xy^2 + x^2y$

$x = at^2$, $y = 2at$ verify by the direct substitution

Given that

$$z = xy^2 + x^2y \qquad x = at^2 \qquad y = 2at$$

$$\frac{\partial z}{\partial x} = y^2(1) + y(2x) \qquad \frac{dx}{dt} = a(2t) \qquad \frac{dy}{dt} = 2a(1)$$

$$\frac{\partial z}{\partial x} = y^2 + 2xy \qquad \frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{\partial z}{\partial y} = x(2y) + x^2(1)$$

$$\frac{\partial z}{\partial y} = 2xy + x^2$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (y^2 + 2xy)\, 2at + (2xy + x^2)\, 2a$$

$$= \left[(2at)^2 + 2(at^2)(2at)\right] 2at + \left(2(at^2)(2at) + (at^2)^2\right) 2a$$

$$= \left[4a^2t^2 + 4a^2t^3\right] 2at + \left[4a^2t^3 + a^2t^4\right] 2a$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$\frac{dz}{dt} = 16a^3t^3 + 10a^3t^4 \quad \#$$

<u>Verification:</u> $z = xy^2 + x^2y$

$$z = (at^2)(2at)^2 + (at^2)^2(2at)$$

$$z = (at^2)(4a^2t^2) + a^2t^4(2at)$$

$$z = 4a^3t^4 + 2a^3t^5$$

$$\frac{dz}{dt} = 4a^3 \cdot \frac{d}{dt}(t^4) + 2a^3 \cdot \frac{d}{dt}(t^5)$$

$$\frac{dz}{dt} = 4a^3(4t^3) + 2a^3 \cdot (5t^4)$$

$$\frac{dz}{dt} = 16a^3t^3 + 10a^3t^4 \quad //$$

2019 (4m)

If $z = x^2+y^2$, $x = at^2$, $y = 2at$

find $\frac{dz}{dt}$

$$z = x^2+y^2$$

$$z = (at^2)^2 + (2at)^2$$

$$z = a^2t^4 + 4a^2t^2$$

$$\frac{dz}{dt} = a^2 \cdot \frac{d}{dt}(t^4) + 4a^2 \cdot \frac{d}{dt}(t^2)$$

$$\frac{dz}{dt} = a^2(4t^3) + 4a^2(2t)$$

$$\frac{dz}{dt} = 4a^2t^3 + 8a^2t.$$

(or)

$z = x^2+y^2$ $\qquad$ $x = at^2$ $\qquad$ $y = 2at$

$\frac{\partial z}{\partial x} = 2x+0$ $\qquad$ $\frac{dx}{dt} = a(2t)$ $\qquad$ $\frac{dy}{dt} = 2a(1)$

$\frac{\partial z}{\partial x} = 2x$ $\qquad$ $\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

$$\frac{\partial z}{\partial y} = 0+2y$$

$$\frac{\partial z}{\partial y} = 2y$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x)(2at) + 2y(2a)$$

$$= 4xat + 4ay$$

$$= 4at(at^2) + 4a(2at)$$

$$= 4a^2t^3 + 8a^2t.$$

$$\frac{dz}{dt} = 4a^2t^3 + 8a^2t. //$$

NOTE:

Let $z = f(x, y)$ and $x = \phi(u, v)$ and $y = \psi(u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Example: z is a function of x and y. Prove that

if $x = e^u + e^{-v}$ $y = e^{-u} - e^v$ then

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \cdot \frac{\partial z}{\partial x} - y \cdot \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} \frac{\partial}{\partial u}\left(e^{u} + e^{-v}\right) + \frac{\partial z}{\partial y} \cdot \frac{\partial}{\partial u}\left(e^{-u} - e^{v}\right)$$

$$= \frac{\partial z}{\partial x}\left[e^{u} + 0\right] + \frac{\partial z}{\partial y}\left[e^{-u}(-1) - 0\right]$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} e^{u} - \frac{\partial z}{\partial y} e^{-u} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x} \frac{\partial}{\partial v}\left(e^{u} + e^{-v}\right) + \frac{\partial z}{\partial y} \frac{\partial}{\partial v}\left(e^{-u} - e^{v}\right)$$

$$= \frac{\partial z}{\partial x}\left[0 + e^{-v}(-1)\right] + \frac{\partial z}{\partial y} \cdot \left(0 - e^{v}\right)$$

$$= -\frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y}\left(-e^{v}\right)$$

$$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x} e^{-v} - \frac{\partial z}{\partial y} e^{v} \quad \text{--- (2)}$$

LHS: $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

$$= \frac{\partial z}{\partial x} e^{u} - \frac{\partial z}{\partial y} e^{-u} - \left[-\frac{\partial z}{\partial x} e^{-v} - \frac{\partial z}{\partial y} e^{v}\right]$$

$\frac{\partial z}{\partial x}e^{u} - \frac{\partial z}{\partial y}e^{-u} + \frac{\partial z}{\partial x}e^{-v} + \frac{\partial z}{\partial y}e^{v}$

$= \frac{\partial z}{\partial x}\left[e^{u} + e^{-v}\right] - \frac{\partial z}{\partial y}\left[e^{-u} - e^{v}\right]$

$= \frac{\partial z}{\partial x}(x) - \frac{\partial z}{\partial y}(y)$

$= x \cdot \frac{\partial z}{\partial x} - y \cdot \frac{\partial z}{\partial y}$

$=$ RHS

LHS $=$ RHS //

Hence proved.

If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$

find $\frac{\partial u}{\partial r}$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial}{\partial x}(x^2 - y^2) \cdot \frac{\partial}{\partial r}(2r - 3s + 4) + \frac{\partial}{\partial y}(x^2 - y^2) \cdot \frac{\partial}{\partial r}(-r + 8s - 5)$$

$$= (2x - 0)(2(1) - 0 + 0) + (0 - 2y) \cdot ((-1) + 0 - 0)$$

$$\frac{\partial u}{\partial r} = 4x + 2y \text{ //}$$

(*) If $z = \frac{\cos y}{x}$ and $x = u^2 - v$, $y = e^v$: find $\frac{\partial z}{\partial v}$

z is a function of x & y

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial}{\partial x}\left(\frac{\cos y}{x}\right) \frac{\partial}{\partial v}\left(u^2 - v\right) + \frac{\partial}{\partial y}\left(\frac{\cos y}{x}\right) \cdot \frac{\partial}{\partial v}\left(e^v\right)$$

$$= \cos y \cdot \frac{\partial}{\partial x}\left(\frac{1}{x}\right) [0-1] + \frac{1}{x} \cdot \frac{\partial}{\partial y}(\cos y) \cdot (e^v)$$

$$= \cos y \left(\frac{-1}{x^2}\right)(-1) + \frac{1}{x}(-\sin y)\, e^v$$

$$\frac{\partial z}{\partial v} = \frac{\cos y}{x^2} - \frac{\sin y \cdot e^v}{x}$$

(*) If $z = \frac{\sin u}{\cos v}$, $u = \frac{\cos y}{\sin x}$, $v = \frac{\cos x}{\sin y}$

find $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial u}\left(\frac{\sin u}{\cos v}\right) \cdot \frac{\partial}{\partial x}\left(\frac{\cos y}{\sin x}\right) + \frac{\partial}{\partial v}\left(\frac{\sin u}{\cos v}\right) \cdot \frac{\partial}{\partial x}\left(\frac{\cos x}{\sin y}\right)$$

$$= \frac{1}{\cos v} \cdot \frac{\partial}{\partial u}(\sin u) \cos y \cdot \frac{\partial}{\partial x}\left(\frac{1}{\sin x}\right) + \sin u \cdot \frac{\partial}{\partial v}\left(\frac{1}{\cos v}\right) \frac{1}{\sin y} \frac{\partial}{\partial x}(\cos x)$$

$$= \frac{-\cos u \cos y}{\cos v} \cdot (\cos x) + \frac{-\sin u}{\cos^2 v} \cdot \frac{1}{\sin y} (\sin x)(\sin v)$$

$$= \frac{-\cos u \cos x \cos y}{\cos v \sin^2 x} - \frac{\sin u \sin x \sin v}{\cos^2 v \cdot \sin y}$$

$$= -\left[\frac{\cos u}{\cos v} \cdot \frac{\cos x}{\sin x} \cdot \frac{\cos y}{\sin x} + \frac{\sin u}{\cos v} \cdot \frac{\sin v}{\cos v} \cdot \frac{\sin x}{\sin y}\right]$$

$$= \left[-\frac{\cos u \cdot u \cdot \cot x}{\cot v} + \frac{z \cdot \sin v \sin x}{\cos v \sin y}\right]$$

$$\frac{\partial z}{\partial x} = -\left[\frac{u\cos u \cot x \sin y + z\sin v \sin x}{\cos v \sin y}\right]$$

(5) $w = x^2+y^2$, $x = r-s$, $y = r+s$, find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial}{\partial x}(x^2+y^2)\cdot\frac{\partial}{\partial r}(r-s) + \frac{\partial}{\partial y}(x^2+y^2)\cdot\frac{\partial}{\partial r}(r+s)$$

$$= (2x)(1) + (2y)(1)$$

$$\frac{\partial w}{\partial r} = 2x+2y$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}(x^2+y^2)\cdot\frac{\partial}{\partial s}(r-s) + \frac{\partial}{\partial y}(x^2+y^2)\cdot\frac{\partial}{\partial s}(r+s)$$

$$= 2x(-1) + 2y(1)$$

$$\frac{\partial w}{\partial s} = 2y - 2x$$

(6) If $z = \log(u^2+v)$, $u = e^{x+y^2}$, $v = x+y^2$ P.T

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial u}\left(\log(u^2+v)\right) \cdot \frac{\partial}{\partial x}\left(e^{x+y^2}\right) + \frac{\partial}{\partial v}\left(\log(u^2+v)\right) \cdot \frac{\partial}{\partial x}\left(x+y^2\right)$$

$$= \frac{1}{u^2+v}(2u) \cdot e^{x+y^2} \cdot (1) + \frac{1}{u^2+v}(1)$$

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v} e^{x+y^2} + \frac{1}{u^2+v} \quad \text{— (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial u}\left(\log(u^2+v)\right) \cdot \frac{\partial}{\partial y}\left(e^{x+y^2}\right) + \frac{\partial}{\partial v}\left(\log(u^2+v)\right) \cdot \frac{\partial}{\partial y}\left(x+y^2\right)$$

$$= \frac{1}{u^2+v} \cdot 2u \cdot e^{x+y^2} \cdot (2y) + \frac{1}{u^2+v}(1)(2y)$$

$$\frac{\partial z}{\partial y} = \frac{4uye^{x+y^2}}{u^2+v} + \frac{2y}{u^2+v} \quad \text{— (2)}$$

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2ue^{x+y^2}}{u^2+v} + \frac{1}{u^2+v} - \frac{4uye^{x+y^2}}{u^2+v} \cdot y - \frac{2y}{u^2+v}$$

$$= \frac{e^{x+y^2}(2u)}{u^2+v}(1-2y) + \frac{1-2y}{u^2+v}$$

$$= \frac{1-2y}{u^2+v}\left[2ue^{x+y^2}+1\right]$$

Example (previous) 2019 4m

Q) If $H = f(y-z, z-x, x-y)$

Then P.T

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$$

Given that,

$$H = f(y-z \quad z-x \quad x-y)$$

$$U = y-z \quad v = z-x \quad w = x-y$$

$$H = f(U, v, w)$$

P.D w.r.to x

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial H}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial H}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= \frac{\partial H}{\partial U} \cdot \frac{\partial}{\partial x}(y-z) + \frac{\partial H}{\partial v} \cdot \frac{\partial}{\partial x}(z-x) + \frac{\partial H}{\partial w} \frac{\partial}{\partial x}(x-y)$$

$$= \frac{\partial H}{\partial U}(0) + \frac{\partial H}{\partial v}(0-1) + \frac{\partial H}{\partial w}(1-0)$$

$$\frac{\partial H}{\partial x} = -\frac{\partial H}{\partial v} + \frac{\partial H}{\partial w} \quad \text{—(1)}$$

$$H = f(U, v, w)$$

P.D w.r.to y

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial H}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial H}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$= \frac{\partial H}{\partial u} \frac{\partial}{\partial y}(y-z) + \frac{\partial H}{\partial v} \frac{\partial}{\partial y}(z-x) + \frac{\partial H}{\partial w} \frac{\partial}{\partial y}(x-y)$$

$$= \frac{\partial H}{\partial u}(1-0) + \frac{\partial H}{\partial v}(0) + \frac{\partial H}{\partial w}(0-1)$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w} \quad \text{---} (2)$$

$$H = f(u, v, w)$$

P.D w.r.t z

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial H}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial H}{\partial w} \cdot \frac{\partial w}{\partial z}$$

$$= \frac{\partial H}{\partial u} \cdot \frac{\partial}{\partial z}(y-z) + \frac{\partial H}{\partial v} \cdot \frac{\partial}{\partial z}(z-x) + \frac{\partial H}{\partial w} \cdot \frac{\partial}{\partial z}(x-y)$$

$$= \frac{\partial H}{\partial u}(-1) + \frac{\partial H}{\partial v}(1-0) + \frac{\partial H}{\partial w}(0)$$

$$\frac{\partial H}{\partial z} = -\frac{\partial H}{\partial u} + \frac{\partial H}{\partial v} \quad \text{---} (3)$$

Adding eq (1) (2) (3)

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = \cancel{-\frac{\partial H}{\partial v}} + \cancel{\frac{\partial H}{\partial w}} + \cancel{\frac{\partial H}{\partial u}} - \cancel{\frac{\partial H}{\partial w}} - \cancel{\frac{\partial H}{\partial u}} + \cancel{\frac{\partial H}{\partial v}}$$

$$= 0$$

Hence proved.

Example:

If H is a homogenous function in terms of x, y & z of degree n then show that

$$x \cdot \frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} + z \cdot \frac{\partial H}{\partial z} = nH.$$

[Euler's theorem for three independent variables x, y, z]

$$\boxed{z = x^n \cdot f\left(\frac{y}{x}\right)}$$

Given that

H is a homogenous function in terms of x, y, z of degree n.

Let, $H = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$

Let, $u = \frac{y}{x}$ $v = \frac{z}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = y \cdot \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = y\left(\frac{-1}{x^2}\right) = \frac{-y}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x} \cdot \frac{\partial}{\partial y}(y) = \frac{1}{x}(1) = \frac{1}{x}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{x}\right) = 0$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{z}{x}\right) = z \cdot \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = \frac{-z}{x^2}.$$

$\frac{\partial}{\partial y}\left(\frac{z}{x}\right) = 0$

$\frac{\partial}{\partial z}\left(\frac{z}{x}\right) = \frac{1}{x} \cdot \frac{\partial}{\partial z}(z) = \frac{1}{x}$

$\underbrace{H = x^n f(U, V)}_{u}$

P.D w.r.t x

$$\frac{\partial H}{\partial x} = x^n \cdot \frac{\partial}{\partial x} f(U, V) + f(U, V) \cdot \frac{\partial}{\partial x}(x^n)$$

$$= x^n \left[\frac{\partial f}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial f}{\partial V} \cdot \frac{\partial V}{\partial x}\right] + f(U, V) \cdot n \cdot x^{n-1}$$

$$= x^n \left[\frac{\partial f}{\partial U}\left(\frac{-y}{x^2}\right) + \frac{\partial f}{\partial V}\left(\frac{-z}{x^2}\right)\right] + f(U, V) \cdot nx^{n-1}$$

$$= \frac{-x^n \cdot y}{x^2} \cdot \frac{\partial f}{\partial U} - \frac{x^n \cdot z}{x^2} \cdot \frac{\partial f}{\partial V} + f(U, V)\, nx^{n-1}$$

$$\frac{\partial H}{\partial x} = -x^{n-2} \cdot y \,\frac{\partial f}{\partial U} - x^{n-2} \cdot z \cdot \frac{\partial f}{\partial V} + nx^{n-1} f(U, V)$$

both sides multiply with x

$$x\frac{\partial H}{\partial x} = -x \cdot x^{n-2} \cdot y \cdot \frac{\partial f}{\partial U} - x \cdot x^{n-2} \cdot z \cdot \frac{\partial f}{\partial V} + x \cdot nx^{n-1} f(U, V)$$

$$x\frac{\partial H}{\partial x} = -x^{n-1} \cdot y \cdot \frac{\partial f}{\partial U} - x^{n-1} \cdot z \cdot \frac{\partial f}{\partial V} + n \cdot x^n f(U, V) \quad \text{——} (1)$$

$$H = x^n \cdot f(u, v)$$

P.D. w.r.to y

$$\frac{\partial H}{\partial y} = x^n \cdot \frac{\partial}{\partial y}\left[f(u,v)\right]$$

$$= x^n \cdot \left[\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}\right]$$

$$= x^n \left[\frac{\partial f}{\partial u}\left(\frac{1}{x}\right) + \frac{\partial f}{\partial v}(0)\right]$$

$$= x^n \cdot \frac{\partial f}{\partial u}\left(\frac{1}{x}\right)$$

$$= \frac{x^n}{x} \cdot \frac{\partial f}{\partial u}$$

$$\frac{\partial H}{\partial y} = x^{n-1} \cdot \frac{\partial f}{\partial u}$$

B.S multiply with y

$$y \cdot \frac{\partial H}{\partial y} = x^{n-1} \cdot y \cdot \frac{\partial f}{\partial u} \quad \text{——— (2)}$$

$$H = x^n \cdot f(u, v)$$

P.D w.r.to z

$$\frac{\partial H}{\partial z} = x^n \frac{\partial}{\partial z}\left(f(u,v)\right)$$

$$= x^n \left[\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z}\right]$$

$$= x^n \left[\frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}\left(\frac{1}{x}\right) \right]$$

$$= \frac{x^n}{x} \cdot \frac{\partial f}{\partial v}$$

$$\frac{\partial H}{\partial z} = x^{n-1} \cdot \frac{\partial f}{\partial v}$$

B.S z on B.S

$$\therefore z\frac{\partial H}{\partial z} = z \cdot x^{n-1} \cdot \frac{\partial f}{\partial v} \quad \text{— (3)}$$

Adding eq (1) (2) (3)

$$x\frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} + z \cdot \frac{\partial H}{\partial z} = \cancel{-x^{n-1} y \cdot \frac{\partial f}{\partial u}} \cancel{- x^{n-1} z \cdot \frac{\partial f}{\partial v}} + n \cdot x^n$$

$$f(u,v) \cancel{+ x^{n-1} y \cdot \frac{\partial f}{\partial u}} \cancel{+ z x^{n-1} \cdot \frac{\partial f}{\partial v}}$$

$$= n \cdot x^n \cdot f(u,v)$$

$$= nH$$

$$\therefore x \cdot \frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} + z \cdot \frac{\partial H}{\partial z} = nH \;//$$

Hence proved.

Implicit functions:

$$f(x, y) = 0$$

$$\frac{\partial y}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

(1) If $x^y = y^x$ find $\frac{dy}{dx}$

Given

$$x^y = y^x$$

$$x^y / y^x \neq 0$$

Apply log on bs

$$\log x^y = \log y^x$$

$$y \log x = x \log y$$

$$y \log x - x \log y = 0$$

Let $f = y \log x - x \log y$

$$\frac{\partial f}{\partial x} = y\left(\frac{1}{x}\right) - \log y \,(1)$$

$$\frac{\partial f}{\partial y} = \log x(1) - x\left(\frac{1}{y}\right)$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$= -\frac{\left[\frac{y}{x} - \log y\right]}{\left[\log x - \frac{x}{y}\right]}$$

$$= -\frac{\left[\frac{y - x\log y}{x}\right]}{\left[\frac{y\log x - x}{y}\right]}$$

$$= \left[\frac{x\log y - y}{x}\right] \cdot \frac{y}{y\log x - x}$$

$$\frac{dy}{dx} = \frac{y}{x}\left(\frac{x\log y - y}{y\log x - x}\right) //$$

If $x\sin(x-y) - (x+y) = 0$ find $\frac{dy}{dx}$

Given,

$$x\sin(x-y) - (x+y) = 0 \quad — ①$$

let, $f = x\sin(x-y) - x - y$

$$\frac{\partial f}{\partial x} = x \cdot \frac{\partial}{\partial x}\left(\sin(x-y)\right) + \sin(x-y) \cdot \frac{\partial}{\partial x}(x) - 1 - 0$$

$$= x\cos(x-y)\cdot\frac{\partial}{\partial x}(x-y)+\sin(x-y)(1)-1$$

$$= x\cos(x-y)(1)+\sin(x-y)-1$$

$$f_x = x\cos(x-y)+\sin(x-y)-1$$

$$f = x\sin(x-y)-x-y$$

$$\frac{\partial f}{\partial y} = x\cdot\frac{\partial}{\partial y}\left(\sin(x-y)\right)-0-1$$

$$= x\cos(x-y)\frac{\partial}{\partial y}(x-y)-1$$

$$= x\cos(x-y)(0-1)-1$$

$$= -x\cos(x-y)-1$$

$$f_y = -\left[x\cos(x-y)+1\right]$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$= -\frac{\left[x\cos(x-y)+\sin(x-y)-1\right]}{-\left[x\cos(x-y)+1\right]}$$

$$= \frac{\left[x\cos(x-y)+\left(\frac{x+y}{x}\right)-1\right]}{\left[x\cos(x+y)+1\right]}$$ [from eq①]

$\sin(x-y) = \frac{x+y}{x}$

$$\frac{\left[\frac{x^2\cos(x-y)+\cancel{x}+y-\cancel{x}}{x}\right]}{[x\cos(x-y)+1]}$$

$$\frac{dy}{dx} = \frac{x^2\cos(x-y)+y}{x[x\cos(x-y)+1]}$$

$$\frac{dy}{dx} = \frac{x^2\cos(x-y)+y}{x^2\cos(x-y)+x}$$

Q) If $(\cos x)^y - (\sin y)^x = 0$ find $\frac{dy}{dx}$

Given $(\cos x)^y - (\sin y)^x = 0$

$$(\cos x)^y = (\sin y)^x$$

Apply log on bs

$$\log(\cos x)^y = \log(\sin y)^x$$

$$y \cdot \log(\cos x) = x \cdot \log(\sin y)$$

$$y \cdot \log(\cos x) - x\log(\sin y) = 0$$

Let,

$$f = y\log(\cos x) - x\log(\sin y)$$

$$\frac{\partial f}{\partial x} = y\frac{1}{\cos x}(-\sin x) - \log(\sin y) \cdot \frac{\partial}{\partial x}(x)$$

$$\frac{\partial f}{\partial x} = \frac{-y\sin x}{\cos x} - \log(\sin y)$$

$$\frac{\partial f}{\partial x} = -\left[y\tan x + \log(\sin y)\right]$$

$$f = y\log(\cos x) - x\log(\sin y)$$

$$\frac{\partial f}{\partial y} = \log(\cos x)\,(1) - x\frac{1}{\sin y}(\cos y)$$

$$\frac{\partial f}{\partial y} = \log(\cos x) - x\cot y$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\frac{dy}{dx} = \frac{y\tan x + \log(\sin y)}{\log(\cos x) - x\cot y} \quad //$$

(4) $y^{x^y} = \sin x$ find $\frac{dy}{dx}$

Given, $y^{x^y} = \sin x$

Apply log on bs

$$\log\left[y^{x^y}\right] = \log(\sin x)$$

$x^y \log y = \log(\sin x)$

$x^y \log y - \log(\sin x) = 0$

Let, $f = x^y \log(y) - \log(\sin x)$

$\frac{\partial f}{\partial x} = \log(y) \cdot y \cdot x^{y-1} - \frac{1}{\sin x} \cdot \cos x$

$f_x = \log y \cdot y x^{y-1} - \cot x$

$f = x^y \log(y) - \log(\sin x)$

$\frac{\partial f}{\partial y} = x^y \cdot \frac{\partial}{\partial y}(\log y) + \log y \cdot \frac{\partial}{\partial y}(x^y) - 0$

$= x^y\left(\frac{1}{y}\right) + \log y \; x^y \log x$

$f_y = \frac{x^y}{y} + \log y \cdot x^y \log x$

$\frac{dy}{dx} = -\frac{f_x}{f_y}$

$= \frac{-[\log y \cdot y x^{y-1} - \cot x]}{\frac{x^y}{y} + \log y \cdot x^y \log x}$

$$= \frac{-\left[y \cdot x^{y-1} \log y - \cot x\right]}{\frac{x^y + x^y \cdot y \log y \log x}{y}}$$

$$= \frac{-y\left[y \cdot x^{y-1} \log y - \cot x\right]}{x^y + x^y \cdot y \log y \log x}$$

$$= \frac{-\left[y^2 \cdot x^{y-1} \log y - y \cot x\right]}{x^y + x^y \cdot y \log y \log x}$$

$$\frac{dy}{dx} = \frac{y \cot x - y^2 x^{y-1} \log y}{x^y + x^y \cdot y \log y \log x} \;//$$

(5) $(\text{Tan}\,x)^y + y^{\cot x} = a$ find $\frac{dy}{dx}$

Given,

$$(\text{Tan}\,x)^y + y^{\cot x} = a$$

$$(\text{Tan}\,x)^y + y^{\cot x} - a = 0$$

Let, $f = (\text{Tan}\,x)^y + y^{\cot x} - a$

$$\frac{\partial f}{\partial x} = y\cdot(\tan x)^{y-1}\cdot\frac{\partial}{\partial x}(\tan x) + y^{\cot x}\cdot\log(y)\cdot\frac{\partial}{\partial x}(\cot x)$$

$$f_x = y(\tan x)^{y-1}(\sec^2 x) + y^{\cot x}\log y(-\operatorname{cosec}^2 x)$$

$$f_x = y(\tan x)^{y-1}(\sec^2 x) - y^{\cot x}\cdot\log y(\operatorname{cosec}^2 x)$$

$$f = (\tan x)^y + y^{\cot x} - a$$

$$\frac{\partial f}{\partial y} = (\tan x)^y\cdot\log(\tan x)\cdot\frac{\partial}{\partial y}(y) + \cot x\cdot y^{\cot x-1}\cdot\frac{\partial}{\partial y}(y)$$

$$f_y = (\tan x)^y\log(\tan x) + \cot x\cdot y^{\cot x-1}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\frac{dy}{dx} = -\frac{\left[y(\tan x)^{y-1}(\sec^2 x) - y^{\cot x}\log y(\operatorname{cosec}^2 x)\right]}{\left[(\tan x)^y\log(\tan x) + \cot x\cdot y^{\cot x-1}\right]} \quad //$$

NOTE: If $f(x,y)=0$ Then

$$\frac{d^2y}{dx^2} = \frac{-\left[f_{x^2}\cdot(f_y)^2 - 2f_{yx}f_x f_y + f_{y^2}(f_x)^2\right]}{(f_y)^3}$$

<u>Example:</u> Prove that if $y^3 - 3ax^2 + x^3 = 0$ then

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$$

Given, $y^3 - 3ax^2 + x^3 = 0$ — (1)

$$y^3 + x^3 = 3ax^2$$

Let, $f(x, y) = y^3 - 3ax^2 + x^3$

$$f_x = \frac{\partial f}{\partial x} = 0 - 3a(2x) + 3x^2 = 3x^2 - 6ax$$

$$f_{x^2} = \frac{\partial^2 f}{\partial x^2} = 3(2x) - 6a(1) = 6x - 6a$$

$$f_y = \frac{\partial f}{\partial y} = 3y^2 - 0 + 0 = 3y^2$$

$$f_{y^2} = \frac{\partial^2 f}{\partial y^2} = 3(2y) = 6y$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 - 6ax\right) = 0$$

$$\frac{d^2y}{dx^2} = \frac{-\left[f_{x^2}(f_y)^2 - 2f_{yx}f_xf_y + f_{y^2}(f_x)^2\right]}{(f_y)^3}$$

$$= \frac{-\left[(6x - 6a)(3y^2)^2 - 2(0)(3x^2 - 6ax)(3y^2) + 6y(3x^2 - 6ax)^2\right]}{(3y^2)^3}$$

$$-\frac{\left[6(x-a)\cdot 9y^4+6y(3)^2(x^2-2ax)^2\right]}{27y^6}$$

$$-\frac{\left[54(x-a)y^4+54y(x^4+4a^2x^2-4ax^3)\right]}{27y^6}$$

$$-\frac{54y\left[(x-a)y^3+x^4+4a^2x^2-4ax^3\right]}{27y^5}$$

$$-\frac{2\left[xy^3-ay^3+x^4+4a^2x^2-4ax^3\right]}{y^5}$$

$$-\frac{2\left[x(y^3+x^3)-ay^3+4a^2x^2-4ax^3\right]}{y^5}$$

from eq(1)

$y^3+x^3=3ax^2$

$$-\frac{2\left[x(3ax^2)-ay^3+4a^2x^2-4ax^3\right]}{y^5}$$

$$-\frac{2\left[3ax^3-ay^3+4a^2x^2-4ax^3\right]}{y^5}$$

$$-\frac{2\left[-ax^3-ay^3+4a^2x^2\right]}{y^5}$$

$$-\frac{2\left[-a(x^3+y^3)+4a^2x^2\right]}{y^5}$$

$$= \frac{-2\left[-a\left(3ax^2\right)+4a^2x^2\right]}{y^5}$$ from eq (1) $y^2+x^2=3ax^2$

$$= \frac{-2\left[-3a^2x^2+4a^2x^2\right]}{y^5}$$

$$\frac{d^2y}{dx^2} = -2\,\frac{a^2x^2}{y^5}$$

$$\frac{d^2y}{dx^2} + 2\,\frac{a^2x^2}{y^5} = 0 \quad //$$

Hence proved

2) If $x^3+y^3=3axy$ find $\frac{d^2y}{dx^2}$

Given, $x^3+y^3=3axy$ — (1)

$x^3+y^3-3axy=0$

let, $f(x,y)=x^3+y^3-3axy$

$$f_x = \frac{\partial f}{\partial x} = 3x^2+0-3ay(1) = 3x^2-3ay$$

$$f_{x^2} = \frac{\partial^2 f}{\partial x^2} = 3(2x)-0 = 6x$$

$$f_y = \frac{\partial f}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

$$f_{y^2} = \frac{\partial^2 f}{\partial y^2} = 3(2y) - 0 = 6y$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 - 3ay\right) = 0 - 3a(1) = -3a$$

$$\frac{d^2y}{dx^2} = \frac{-\left[f_{x^2}(f_y)^2 - 2f_{yx}\, f_x f_y + f_{y^2}(f_x)^2\right]}{(f_y)^3}$$

$$= \frac{-\left[6x(3y^2 - 3ax)^2 - 2(-3a)(3x^2 - 3ay)(3y^2 - 3ax) + 6y(3x^2 - 3ay)^2\right]}{(3y^2 - 3ax)^3}$$

$$= \frac{-\left[6x(9)\left[y^2 - ax\right]^2 + 6a\left[9x^2y^2 - 9ax^3 - 9ay^3 + 9a^2xy\right] + 6y(9)\left(x^4 + a^2y^2 - 2ax^2y\right)\right]}{(3)^3(y^2 - ax)^3}$$

$$= \frac{-\left[54x\left[y^4 + a^2x^2 - 2axy^2\right] + 54a\left[x^2y^2 - ax^3 - ay^3 + a^2xy\right] + 54y\left[x^4 + a^2y^2 - 2ax^2y\right]\right]}{27(y^2 - ax)^3}$$

$$= \frac{-\overset{2}{\cancel{54}}\left[\underline{xy^4} + \cancel{a^2x^3} - 2ax^2y^2 + ax^2y^2 - \cancel{a^2x^3} - \cancel{a^2y^3} + a^3xy + \underline{x^4y} + \cancel{a^2y^3} - 2ax^2y^2\right]}{\cancel{27}(y^2 - ax)^3}$$

$$= \frac{-2[xy[x^3+y^3]-3ax^2y^2+a^3xy]}{(y^2-ax)^3}$$

$$= \frac{-2[xy[3axy]-3ax^2y^2+a^3xy]}{(y^2-ax)^3}$$

from eq ①
$x^3+y^3=3axy$

$$= \frac{\cancel{-}2[\cancel{3ax^2y^2}-\cancel{3ax^2y^2}+a^3xy]}{\cancel{-}(ax-y^2)^3}$$

$$= \frac{2a^3xy}{(ax-y^2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax-y^2)^3}$$

(*) $x^4+y^4=4a^2xy$ find $\frac{d^2y}{dx^2}$

let $f(x,y) = x^4+y^5-4a^2xy$

$f_x = 4x^3-4a^2y$

$f_x = 4(x^3-a^2y)$

$f_{x^2} = 4[3x^2-0]$

$f_{x^2} = 12x^2$

$f_y = 4y^3 - 4a^2x$

$f_y = 4(y^3 - a^2x)$

$f_{y^2} = 4[3y^2 - 0]$

$f_{y^2} = 12y^2$

$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$= -4a^2$

$$\frac{d^2y}{dx^2} = \frac{-\left[f_{x^2}(f_y)^2 - 2f_{yx}f_xf_y + f_{y^2}(f_x)^2\right]}{(f_y)^3}$$

$$= \frac{-\left\{12x^2\left[4(y^3-a^2x)\right]^2 - 2(-4a^2)16(x^3-a^2y)(y^3-a^2x) + 12y^2\left[4(x^3-a^2y)\right]^2\right\}}{\left[4(y^3-a^2x)\right]^3}$$

$$= \frac{\cancel{-}\left[12x^2\left(16(y^6+a^4x^2-2a^2xy^3)\right) + 8a^2(16)\left[x^3y^3 - a^2x^4 - a^2y^4 + a^4xy\right] + 12\cdot(y)^2 16\left[x^6+a^4y^2-2x^3a^2y\right]\right]}{\cancel{-}64\left[a^2x - y^3\right]^3}$$

$$\frac{d^2y}{dx^2} = \frac{192\{x^2y^6 + a^4x^4 - 2a^2x^3y^3) + 128(a^2x^3y^3 - a^4x^4 - a^4y^4 + a^6xy) + 192(x^6y^2 + a^4y^4 - 2a^2x^3y^3)}{64(a^2x - y^3)^3}$$

$$= \frac{\cancel{64}\left[3(x^2y^6+a^4x^4-2a^2x^3y^3)+2(a^2x^3y^3-a^4x^4-a^4y^4+a^6xy) + 3(x^6y^2+a^4y^4-2a^2x^3y^3)\right.}{\cancel{64}(a^2x-y^3)^3}$$

$$= \frac{\underline{3x^2}y^6+\underline{3a^4}x^4-\underline{6a^2x^3}y^3+2a^2x^3y^3-\underline{2a^4x^4}-\underline{2a^4}y^4+2a^6xy + \underline{3x^6}y^2+\underline{3a^4}y^4-\underline{6a^2x^3}y^3}{(a^2x-y^3)^3}$$

$$= \frac{3x^2y^2(y^4+x^4]+3a^4(x^4+y^4)-12a^2x^3y^3-2a^4(x^4+y^4)+2a^2x^3y^3+2a^6xy}{(a^2x-y^3)^3}$$

$$= \frac{3x^2y^2(4a^2xy)+3a^4(4a^2xy)-10a^2x^3y^3-2a^4(4a^2xy)+2a^6xy}{(a^2x-y^3)^3}$$

$$= \frac{12a^2x^3y^3+12a^6xy-10a^2x^3y^3-8a^6xy+2a^6xy}{(a^2x-y^3)^3}$$

$$= \frac{12a^2x^3y^3+4a^6xy-10a^2x^3y^3+2a^6xy}{(a^2x-y^3)^3}$$

$$= \frac{2a^2x^3y^3 + 6a^6xy}{(a^2x - y^3)^3}$$

$$\frac{d^2y}{dx^2} = \frac{2a^2xy(a^2x^2y^2 + 3a^4)}{(a^2x - y^3)^3}$$ //

(2) $x^5 + y^5 = 5a^3xy$ find $\frac{d^2y}{dx^2}$

$$x^5 + y^5 = 5a^3xy \quad \text{— ①}$$

let $f(x,y) = x^5 + y^5 - 5a^3xy$

$$f_x = 5x^4 - 5a^3y$$

$$f_{x^2} = 20x^3$$

$$f_y = 5y^4 - 5a^3x$$

$$f_{y^2} = 20y^3$$

$$f_{yx} = \frac{\partial}{\partial y}(5x^4 - 5a^3y)$$

$$= -5a^3$$

$$\frac{d^2y}{dx^2} = \frac{-[f_{x^2}(f_y)^2 - 2f_{yx}f_xf_y + f_{y^2}(f_x)^2]}{(f_y)^3}$$

$$= \frac{-\left[20x^3(5y^4-5a^3x)^2 - 2(-5a^3)(5x^4-5a^3y)(5y^4-5a^3x) + 20y^3(5x^4-5a^3y)^2\right]}{(5y^4-5a^3x)^3}$$

$$= \frac{-\left[20x^3(5)^2[y^4-a^3x]^2 + 10a^3(25)(x^4-a^3y)(y^4-a^3x) + 20y^3(25)[x^4-a^3y]^2\right]}{25\times5\,(y^4-a^3x)^3}$$

$$= \frac{-\cancel{25\times5}\left[4x^3(y^8+a^6x^2-2y^4a^3x) + 2a^3(x^4y^4-a^3x^5-a^3y^5+a^6xy) + 4y^3(x^8+a^6y^2-2x^4a^3y)\right]}{\cancel{25\times5}(y^4-a^3x)^3}$$

$$= \frac{\cancel{-}\left[4x^3y^8+4a^6x^5-8y^4a^3x^4+2a^3x^4y^4-2a^6x^5-2a^6y^5+2a^9xy + 4y^3x^8+4a^6y^5-8x^4a^3y^4\right]}{\cancel{-}(a^3x-y^4)^3}$$

$$= \frac{4x^3y^3(x^5+y^5)+4a^6(x^5+y^5)-6y^4a^3x^4-2a^6(x^5+y^5)+2a^9xy-8x^4a^3y^4}{(a^3x-y^4)^3}$$

$$\frac{4x^3y^3(5a^3xy)+4a^6(5a^3xy)-14a^3x^4y^4-2a^6(5a^3xy)+2a^9xy}{(a^3x+y^4)^3}$$

$$=\frac{20x^4y^4a^3+20a^9xy-14a^3x^4y^4-10a^9xy+2a^9xy}{(a^3x-y^4)^3}$$

$$=\frac{6a^3x^4y^4+10a^9xy+2a^9xy}{(a^3x-y^4)^3}$$

$$=\frac{6a^3x^4y^4+12a^9xy}{(a^3x-y^4)^3}$$

$$\frac{d^2y}{dx^2}=\frac{6a^3xy(2a^6+x^3y^3)}{(a^3x-y^4)^3}$$

6) If $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a$ find S.T

$$\frac{d^2y}{dx^2}=\frac{-a}{(1-x^2)^{3/2}}$$

Given that

$$x\sqrt{1-y^2}+y\sqrt{1-x^2}=a \quad -①$$

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} - a = 0$$

let $f(x,y) = x\sqrt{1-y^2} + y\sqrt{1-x^2} - a$

P.D w.r.t x

$$f_x = \sqrt{1-y^2}\,\frac{\partial}{\partial x}(x) + y \cdot \frac{\partial}{\partial x}\left(\sqrt{1-x^2}\right) - 0$$

$$= \sqrt{1-y^2}\,(1) + y \cdot \frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x)$$

$$= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$f_x = \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-x^2}}$$

$$f = x\sqrt{1-y^2} + y\sqrt{1-x^2} - a$$

P.D w.r.t y

$$f_y = x\,\frac{\partial}{\partial y}\left(\sqrt{1-y^2}\right) + \sqrt{1-x^2}\,\frac{\partial}{\partial y}(y) - 0$$

$$= x \cdot \frac{1}{\cancel{2}\sqrt{1-y^2}}(-\cancel{2}y) + \sqrt{1-x^2}\,(1)$$

$$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$f_y = \frac{-xy + \sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-y^2}}$$

$$f_y = \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-y^2}}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$= \frac{-\left(\frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-x^2}}\right)}{\frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-y^2}}}$$

$$= -\left[\frac{\cancel{\sqrt{1-x^2}\sqrt{1-y^2} - xy}}{\sqrt{1-x^2}}\right] \cdot \left[\frac{\sqrt{1-y^2}}{\cancel{\sqrt{1-x^2}\sqrt{1-y^2} - xy}}\right]$$

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \quad \text{--- (2)}$$

Derivate w.r.to x

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{1-x^2} \cdot \frac{d}{dx}\left(\sqrt{1-y^2}\right) - \left(\sqrt{1-y^2}\right) \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)}{\left(\sqrt{1-x^2}\right)^2}\right]$$

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{1-x^2}\cdot\frac{1}{\cancel{2}\sqrt{1-y^2}}(-\cancel{2}y)\cdot\frac{dy}{dx} - \sqrt{1-y^2}\cdot\frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x)}{(1-x^2)}\right]$$

$$= -\left[\frac{\frac{-y\sqrt{1-x^2}}{\sqrt{1-y^2}}\left(\frac{dy}{dx}\right) + \sqrt{1-y^2}\,\frac{x}{\sqrt{1-x^2}}}{(1-x^2)}\right]$$

$$= -\left[\frac{\frac{-y\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-y^2}}}\left(\frac{-\cancel{\sqrt{1-y^2}}}{\cancel{\sqrt{1-x^2}}}\right) + \sqrt{1+y^2}\,\frac{x}{\sqrt{1-x^2}}}{(1-x^2)}\right]$$

$$= \frac{-\left[y + \frac{x\sqrt{1-y^2}}{\sqrt{1-x^2}}\right]}{1-x^2}$$

$$= \frac{-\left[\frac{y\cdot\sqrt{1-x^2} + x\cdot\sqrt{1-y^2}}{\sqrt{1-x^2}}\right]}{1-x^2}$$

$$= -\frac{y\sqrt{1-x^2} + x\sqrt{1-y^2}}{\sqrt{1-x^2}\,(1-x^2)}$$ from eq①

$$= \frac{-a}{(1-x^2)^{1/2}(1-x^2)}$$

$\left[y\sqrt{1-x^2}+x\sqrt{1-y^2}=a\right]$

$$= \frac{-a}{(1-x^2)^{1/2+1}}$$

$$\frac{d^2y}{dx^2} = \frac{-a}{(1-x^2)^{3/2}} \quad //$$

Hence proved.

Q. If $f(x,y)=0$ and $\cancel{\phi(y,z)} = \phi(y,z)=0$ Then S.T

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

Given that

$f(x,y)=0$ and $\phi(y,z)=0$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \; -① \quad , \quad \frac{dz}{dy} = \frac{-\phi_y}{\phi_z} \; -②$$

eq② × ①

$$\frac{\cancel{dy}}{dx} \cdot \frac{dz}{\cancel{dy}} = \frac{-f_x}{f_y} \times \frac{-\phi_y}{\phi_z}$$

$$\frac{dz}{dx} = \frac{f_x}{f_y} \cdot \frac{\phi_y}{\phi_z}$$

$$\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \cdot \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} \; //$$

Hence proved.

Example

(7) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. Then show

that $\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$

Sol Given that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{—(1)}$$

Let $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$f_x = a(2x) + 2hy(1) + 0 + 2g(1) + 0 + 0$$

$$f_x = 2ax + 2hy + 2g$$

$$f_x = 2(ax + hy + g)$$

$f_{x^2} = 2[a(1) + 0 + 0)$ $\quad f_{x^2} = 2a$

$$f_y = 0 + 2hx(1) + b(2y) + 0 + 2f(1) + 0$$

$$f_y = 2hx + 2by + 2f$$

$$f_y = 2(hx + by + f)$$

$$f_{y^2} = 2[0 + b(1) + 0]$$

$$f_{y^2} = 2b$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2(ax+hy+g)\right) = 2[a(0) + h(1) + 0)$$

$$= 2(h)$$

$$f_{yx} = 2h$$

$$\frac{d^2y}{dx^2} = \frac{-[f_{x^2}(f_y)^2 - 2f_{yx}f_x f_y + f_{y^2}(f_x)^2]}{(f_y)^3}$$

$$= \frac{-[2a(2(hx+by+f))^2 - 2(2h)2(ax+hy+g)2(hx+by+f) + 2b(2(ax+hy+g))^2}{(2(hx+by+f))^3}$$

$$= \frac{-[2a(4)[hx+by+f]^2 - 16h(ax+hy+g)(hx+by+f) + 2b(4)(ax+hy+g]^2}{8(hx+by+f)^3}$$

$$= \frac{-[8a(hx+by+f)^2 - 16h(ahx^2 + abxy + afx + h^2xy + bhy^2 + hfy + ghx + gby + gf) + 8b(ax+hy+g)^2}{8(hx+by+f)^3}$$

$$= \frac{\cancel{-8}\left[a\left(h^2x^2+b^2y^2+f^2+2bhxy+2bfy+2fhx\right) - 2h\left(ahx^2+abxy+afx+h^2xy+bhy^2+hfy+ghx+gby+gf\right) + b\left[a^2x^2+h^2y^2+g^2+2ahxy+2hgy+2gax\right]\right]}{\cancel{8}\left(hx+by+f\right)^3}$$

$$= \frac{-ah^2x^2-ab^2y^2-af^2-2abhxy-2abfy-2afhx+2ah^2x^2+2habxy+2hafx+2h^3xy+2h^2by^2+2h^2fy+2h^2gx+2hgby+2hgf-a^2bx^2-bh^2y^2-bg^2-2abhxy-2bhgy-2bgax}{(hx+by+f)^3}$$

$$= \frac{-ab\left[by^2+2fy+ax^2+2hxy+2gx\right]+h^2\left[-ax^2+2ax^2+2hxy+2by^2+2fy+2gx-by^2\right]-af^2+2hgf-bg^2}{(hx+by+f)^3}.$$

$$= \frac{-ab\left[ax^2+2hxy+by^2+2fy+2gx\right]+h^2\left[ax^2+2hxy+by^2+2fy+2gx\right]-af^2+2hgf-bg^2}{(hx+by+f)^3}$$

$$= \frac{-ab[-c]+h^2[-c]-af^2+2hgf-bg^2}{(hx+by+f)^3} \quad \text{[from eq ①]}$$

$$= \frac{abc - ch^2 - af^2 + 2hgf - bg^2}{(hx + by + f)^3}$$

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$$

Hence proved //

formulas:

1. $\sin(180 - \theta) = \sin\theta$
2. $\cos(180 - \theta) = -\cos\theta$
3. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
4. $\sin 2\theta = 2\sin\theta\cos\theta$
5. $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Example: If A, B, C are angles of a triangle and $\sin^2 A + \sin^2 B + \sin^2 C = \text{constant}$. Then show that

$$\frac{dA}{dB} = \frac{\tan B - \tan C}{\tan C - \tan A}$$

Given that

A, B, C are angles of a triangle

$$\therefore A + B + C = 180^\circ$$

Given that, $\sin^2 A + \sin^2 B + \sin^2 C = \text{const}.$

$$\sin^2 A + \sin^2 B + \sin^2(180 - (A+B)) = \text{const}$$

$$\sin^2 A + \sin^2 B + \sin^2(A+B) = \text{const}$$

$$\sin^2 A + \sin^2 B + \sin^2(A+B) - \text{const} = 0$$

let, $f = \sin^2 A + \sin^2 B + \sin^2(A+B) - \text{const}$

$$f_A = \frac{\partial f}{\partial A} = 2\sin A\cos A + 0 + 2\sin(A+B)\cdot\cos(A+B) - 0$$

$$f_B = \frac{\partial f}{\partial B} = 0 + 2\sin B\cos B + 2\sin(A+B)\cdot\cos(A+B) - 0$$

$$\frac{dA}{dB} = \frac{-f_B}{f_A}$$

$$= \frac{-[2\sin B\cos B + 2\sin(A+B)\cos(A+B)]}{2\sin A\cos A + 2\sin(A+B)\cos(A+B)}$$

$$= \frac{-[\sin 2B + 2\sin(180-C)\cos(180-C)]}{\sin 2A + 2\sin(180-C)\cos(180-C)}$$

$$= \frac{-[\sin 2B + 2\sin C(-\cos C)]}{\sin 2A + 2\sin C(-\cos C)}$$

$$= \frac{-\left[\sin 2B - 2\sin C\cos C\right]}{\sin 2A - 2\sin C\cos C}$$

$$= \frac{-\left[\sin 2B - \sin 2C\right]}{\sin 2A - \sin 2C}$$

$$= \frac{-\left[\cancel{2}\cos\left(\frac{2B+2C}{2}\right)\sin\left(\frac{2B-2C}{2}\right)\right]}{\cancel{2}\cos\left(\frac{2A+2C}{2}\right)\sin\left(\frac{2A-2C}{2}\right)}$$

$$= -\frac{\cos(B+C)\sin(B-C)}{\cos(A+C)\sin(A-C)}$$

$$= \frac{-\left[\cos(180-A)\sin(B-C)\right]}{\cos(180-B)\sin(A-C)}$$

$$= \frac{-\left[-\cos A\sin(B-C)\right]}{-\cos B\sin(A-C)}$$

$$= \frac{\cos A\sin(B-C)}{-\cos B\sin(A-C)}$$

$$= \frac{\cos A\left[\sin B\cos C - \cos B\sin C\right]}{-\cos B\left[\sin A\cos C - \cos A\sin C\right]}$$

$$= \frac{\cos A \sin B \cos C - \cos A \cos B \sin C}{-\cos B \sin A \cos C + \cos A \cos B \sin C}$$

each term multiply with $\cos A \cos B \cos C$

$$= \frac{\dfrac{\cancel{\cos A} \sin B \cancel{\cos C}}{\cancel{\cos A} \cos B \cancel{\cos C}} - \dfrac{\cancel{\cos A} \cancel{\cos B} \sin C}{\cancel{\cos A} \cancel{\cos B} \cos C}}{\dfrac{-\cancel{\cos B} \sin A \cancel{\cos C}}{\cos A \cancel{\cos B} \cancel{\cos C}} + \dfrac{\cancel{\cos A} \cancel{\cos B} \sin C}{\cancel{\cos A} \cancel{\cos B} \cos C}}$$

$$= \frac{\tan B - \tan C}{-\tan A + \tan C}$$

$$\therefore \frac{dA}{dB} = \frac{\tan B - \tan C}{\tan C - \tan A}$$

Hence proved //

(4) If $z = xy \cdot f\left(\frac{y}{x}\right)$ and z is a constant then

S.T $$\frac{f'(y/x)}{f(y/x)} = \frac{x\left[y + x \cdot \frac{dy}{dx}\right]}{y\left[y - x \cdot \frac{dy}{dx}\right]}$$

Given that, $z = xy \cdot f(y/x)$ [z is const]

Derivate unto x

$$0 = xy \cdot \frac{d}{dx} f\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) \cdot \frac{d}{dx}(xy)$$

$$0 = xy \cdot f'\left(\frac{y}{x}\right) \frac{d}{dx}\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right)\left[x \cdot \frac{dy}{dx} + y(1)\right]$$

$$0 = xy \cdot f'\left(\frac{y}{x}\right)\left[\frac{x \cdot \frac{dy}{dx} - y(1)}{x^2}\right] + f\left(\frac{y}{x}\right)\left[x \cdot \frac{dy}{dx} + y\right]$$

$$f'\left(\frac{y}{x}\right) \cancel{x}y \frac{\left[x \cdot \frac{dy}{dx} - y\right]}{x^2} = -f\left(\frac{y}{x}\right)\left[x \cdot \frac{dy}{dx} + y\right]$$

$$\frac{f'\left(\frac{y}{x}\right)}{f(y/x)} = \frac{-\left[x \frac{dy}{dx} + y\right]}{\frac{y\left(x \cdot \frac{dy}{dx} - y\right)}{x}}$$

$$\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{-x\left[x \cdot \frac{dy}{dx} + y\right]}{y\left[x \cdot \frac{dy}{dx} - y\right]}$$

$$= \frac{\cancel{-}x\left[x \cdot \frac{dy}{dx} + y\right]}{\cancel{-}y\left[y - x \cdot \frac{dy}{dx}\right]}$$

$$\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left(y + x \cdot \frac{dy}{dx}\right)}{y\left(y - x \cdot \frac{dy}{dx}\right)}$$

Hence proved.

Taylor's Theorem for two Variables:

Expand the function $f(x,y)$ in powers of $x-a$ and $y-b$.

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)^2\cdot\frac{\partial^2}{\partial x^2} + 2(x-a)(y-b)\cdot\frac{\partial^2}{\partial x\partial y} + (y-b)^2\cdot\frac{\partial^2}{\partial y^2}\right]f(a,b) +$$
$$\frac{1}{3!}\left[(x-a)^3\cdot\frac{\partial^3}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3}{\partial x^2\partial y} + 3(x-a)(y-b)^2\cdot\frac{\partial^3}{\partial x\partial y^2} + (y-b)^3\cdot\frac{\partial^3}{\partial y^3}\right] f(a,b) + \dots\dots$$

Ex-1. Expand the function $f(x,y) = x^2+xy-y^2$ in powers of $x-1$ and $y+2$.

Sol:- Given $f(x,y) = x^2+xy-y^2$; $x-1$; $y+2$

$a = 1$; $b = -2$

$\Rightarrow \frac{\partial f}{\partial x} = 2x+y(1)-0 = 2x+y$; $\frac{\partial f}{\partial y} = 0+x-2y = x-2y$

$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 2+0 = 2$; $\frac{\partial^2 f}{\partial y^2} = 0-2 = -2$

$\Rightarrow \frac{\partial^3 f}{\partial x^3} = 0$; $\frac{\partial^3 f}{\partial y^3} = 0$

$\Rightarrow \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x-2y) = 1-0 = 1$

$\Rightarrow \frac{\partial^3 f}{\partial x^2\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x\partial y}\right] = \frac{\partial}{\partial x}(1) = 0$

$\Rightarrow \frac{\partial^3 f}{\partial x\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y^2}\right] = \frac{\partial}{\partial x}(-2) = 0$

At $(1,-2)$

$f(1,-2) = 1+(-2)-(-2)^3 \Rightarrow 1-2-4 = -5$

$\frac{\partial f}{\partial x} = 2(1)-2 = 0$; $\frac{\partial f}{\partial y} = 1-2(-2) = 5$; $\frac{\partial^2 f}{\partial x\partial y} = 1$

$\frac{\partial^2 f}{\partial x^2} = 2$; $\frac{\partial^2 f}{\partial y^2} = -2$; $\frac{\partial^3 f}{\partial x^2\partial y} = 0$

$\frac{\partial^3 f}{\partial x^3} = 0$; $\frac{\partial^3 f}{\partial y^3} = 0$; $\frac{\partial^3 f}{\partial x\partial y^2} = 0$

Taylor's Formulae :

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) +$$

$$\frac{1}{2!}\left[(x-a)^2\frac{\partial^2}{\partial x^2} + 2(x-a)(y-b)\cdot\frac{\partial^2}{\partial x\partial y} + (y-b)^2\frac{\partial^2}{\partial y^2}\right] f(a,b) +$$

$$\frac{1}{3!}\left[(x-a)^3\frac{\partial^3}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3}{\partial x^2\partial y} + 3(x-a)(y-b)^2\frac{\partial^3}{\partial x\partial y^2} + (y-b)^3\cdot\frac{\partial^3}{\partial y^3}\right] f(a,b) + \dots .$$

$$\Rightarrow f(x,y) = f(1,-2) + \left[(x-1)\frac{\partial}{\partial x} + (y+2)\frac{\partial}{\partial y}\right] f(1,-2) +$$

$$\frac{1}{2}\left[(x-1)^2\frac{\partial^2}{\partial x^2} + 2(x-1)(y+2)\frac{\partial^2}{\partial x\partial y} + (y+2)^2\frac{\partial^2}{\partial y^2}\right] f(1,-2)$$

$$+\frac{1}{6}\left[(x-1)^1\frac{\partial^3}{\partial x^3} + 3(x-1)^2(y+2)\frac{\partial^3}{\partial x^2\partial y} + 3(x-1)(y+2)^3\frac{\partial^3}{\partial x\partial y^2} + (y+2)^3\frac{\partial^3}{\partial y^3}\right] f(1,-2)$$

$$\Rightarrow f(x,y) = -5 + \left[(x-1)\,0 + (y+2)\,5\right] + \frac{1}{2}\left[(x-1)^2(2) + 2(x-1)(y+2)(1) + (y+2)^2(-2)\right] + \frac{1}{6}(0)$$

$$\Rightarrow f(x,y) = -5 + 5(y+2) + \frac{1}{\cancel{2}}(x-1)^2(\cancel{2}) + \frac{1}{\cancel{2}}\cdot\cancel{2}(x-1)(y+2) + \frac{1}{\cancel{2}}(y+2)^2(-\cancel{2})$$

$$\Rightarrow f(x,y) = (x-1)^2 + (x-1)(y+2) - (y+2)^2 + 5(y+2) - 5$$

2. Expand $x^2y + 3y - 2$ in powers of $x-1$ and $y+2$.

Sol: Given $x^2y + 3y - 2 = f(x,y)$; $x-1$; $y+2$

$a = 1$, $b = -2$

$\frac{\partial f}{\partial x} = 2x(y) + y(0) = 0 = 2xy$ $\frac{\partial f}{\partial y} = x^2(1) + 3(1) - 0 = x^2+3$

$\frac{\partial^2 f}{\partial x^2} = 2(y)(1) = 2y$ $\frac{\partial^2 f}{\partial y^2} = 0 + 0 = 0$

$\frac{\partial^3 f}{\partial x^3} = 0$ $\frac{\partial^3 f}{\partial y^3} = 0$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left[x^2+3\right] = 2x$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right] = \frac{\partial}{\partial x}(2x) = 2$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y^2}\right] = \frac{\partial}{\partial x}(0) = 0$$

at $(1,-2)$ $\quad f(1,-2) = (1)^2(-2)+3(-2)-2 = -2-6-2 = -10$

$$\frac{\partial f}{\partial x} = 2xy = 2(1)(-2) = -4 \quad ; \quad \frac{\partial f}{\partial y} = (1)^2+3 = 4$$

$$\frac{\partial^2 f}{\partial x^2} = 2y = 2(-2) = -4 \quad ; \quad \frac{\partial^2 f}{\partial y^2} = 0 \quad ; \quad \frac{\partial^3 f}{\partial x^2 \partial y} = (2)$$

$$\frac{\partial^3 f}{\partial x^3} = 0 \quad ; \quad \frac{\partial^2 f}{\partial x \partial y} = 2(1) = 2 \quad ; \quad \frac{\partial^3 f}{\partial y^3} = 0 \quad ; \quad \frac{\partial^3 f}{\partial x \partial y^3} = 0$$

Taylor's Formulae :

$$f(x,y) = f(a,b) + \left[(x-a)\cdot\frac{\partial}{\partial x} + (y-b)\cdot\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)^2\cdot\frac{\partial^2}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2}{\partial x \partial y} + (y-b)^2\cdot\frac{\partial^2}{\partial y^2}\right] f(a,b) +$$
$$\frac{1}{3!}\left[(x-a)^3\cdot\frac{\partial^3}{\partial x^3} + 3(x-a)^2(y-b)\cdot\frac{\partial^3}{\partial x^2 \partial y} + 3(x-a)(y-b)^2\cdot\frac{\partial^3}{\partial x \partial y^3} + (y-b)^3\frac{\partial^3}{\partial y^3}\right] f(a,b)$$

$$\Rightarrow f(x,y) = f(1,2) + \left[(x-1)\cdot\frac{\partial}{\partial x} + (y+2)\frac{\partial}{\partial y}\right] f(1,-2) + \frac{1}{2}$$
$$\left[(x-1)^2\frac{\partial^2}{\partial x^2} + 2(x-1)(y+2)\cdot\frac{\partial^2}{\partial x \partial y} + (y+2)^2\frac{\partial^2}{\partial y^2}\right] f(1,-2) + \frac{1}{3}$$
$$\left[(x-1)^3\frac{\partial^3}{\partial x^2} + 3(x-1)^2(y+2)\cdot\frac{\partial^3}{\partial x^2 \partial y} + 3(x-1)(y+2)^2\frac{\partial^3}{\partial x \partial y^3} + 0\right]$$

$$\Rightarrow f(x,y) = -10 + \left[(x-1)(-4) + 4(y+2)\right] + \frac{1}{\cancel{2}}\overset{2}{-\cancel{4}}(x-1)^2 + \frac{1}{\cancel{2}}\cancel{2}(2)(x-1)(y+2)$$
$$+ \frac{1}{2}(y+2)^2(0) + \frac{1}{3}\left((x-1)^3(0)\right) + \frac{1}{\cancel{3}}\cdot\cancel{3}(x-1)^2(y+2)(2) + 0$$

$$\Rightarrow f(x,y) = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2)$$
$$+2(x-1)^2(y+2)$$

3. Expand $f(x,y) = x^2 + xy + y^2$ in powers of $x-2$ & $y-3$

Sol: Given $f(x,y) = x^2 + xy + y^2$; $x-2$; $y-3$; $a=2$; $b=3$

$\frac{\partial f}{\partial x} = 2x + 1(y) + 0 = 2x+y$; $\frac{\partial f}{\partial y} = 0 + x(1) + 2y = 2y+x$

$\frac{\partial^2 f}{\partial x^2} = 2(1) = 2$; $\frac{\partial^2 f}{\partial y^2} = 2(1)+0 = 2$

$\frac{\partial^3 f}{\partial x^3} = 0$; $\frac{\partial^3 f}{\partial y^3} = 0$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}(2y+x) = 0+1 = 1$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right] = \frac{\partial}{\partial x}(2y) = 0$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial x}(2) = 0$$

At $f(2,3) = (2)^2 + (2)(3) + (3)^2 = 4+6+9 = 19$

$\frac{\partial f}{\partial x} = 2(2)+3 = 7$; $\frac{\partial f}{\partial y} = 2(3)+2 = 8$; $\frac{\partial^2 f}{\partial x \partial y} = 1$

$\frac{\partial^2 f}{\partial x^2} = 2$; $\frac{\partial^2 f}{\partial y^2} = 2$; $\frac{\partial^3 f}{\partial x^2 \partial y} = 0$

$\frac{\partial^3 f}{\partial x^3} = 0$; $\frac{\partial^3 f}{\partial y^3} = 0$; $\frac{\partial^3 f}{\partial x \partial y^2} = 0$

Taylor's theorem:

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] \cdot f(a,b) + \frac{1}{2!}\left[(x-a)^2 \cdot \frac{\partial^2}{\partial x^2} + 2(y-b)(x-a) \cdot \frac{\partial^2}{\partial x \partial y} + (y-b)^2 \frac{\partial^2}{\partial y^2}\right] f(a,b) + \frac{1}{3!}\left[(x-a)^3 \frac{\partial^3}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3}{\partial x^2 \partial y} + 3(x-a)(y-b)^2 \frac{\partial^2}{\partial x \partial y^2} + (y-b)^3 \cdot \frac{\partial^3}{\partial y^3}\right] f(a,b)$$

$$\Rightarrow f(x,y) = f(2,3) + \left[(x-2)\frac{\partial}{\partial x} + (y-3)\frac{\partial}{\partial y}\right] f(2,3) + \frac{1}{2}\left[(x-2)^2 \frac{\partial^2}{\partial x^2} + 2(x-2)(y-3) \cdot \frac{\partial^2}{\partial x \partial y} + (y-3)^2 \frac{\partial^2}{\partial y^2}\right] f(2,3) + \frac{1}{3}\left[(x-2)^3 \frac{\partial^3}{\partial x^3} + 3(x-2)^2(y-3)\frac{\partial^3}{\partial x^2 \partial y} + 3(x-2)(y-3)^2 \cdot \frac{\partial^3}{\partial x \partial y^2} + (y-3)^3 \cdot \frac{\partial^3}{\partial y^3}\right] f(2,3)$$

$$\Rightarrow f(x,y) = 19 + 7(x-2) + 8(y-3) + \frac{1}{\cancel{2}} \cdot \cancel{2}(x-2)^2 + \frac{1}{\cancel{2}}\cancel{2}(x-2)(y-3) + \frac{1}{\cancel{2}} \cdot \cancel{2}(y-3)^2 + \frac{1}{3}(0)$$

$$\Rightarrow f(x,y) = 19 + 7(x-2) + 8(y-3) + (x-2)^2 + (x-2)(y-3) + (y-3)^2$$

4. Obtain Taylor's theorem for the functions e^{x+y} at $(0,0)$ for $n=3$.

Sol: Given $f(x,y) = e^{x+y}$; $a=0$; $b=0$ at $n=3$

$\frac{\partial f}{\partial x} = e^{x+y}$; $\frac{\partial f}{\partial y} = e^{x+y}$; $\frac{\partial^2 f}{\partial x \partial y} = e^{x+y}$

$\frac{\partial^2 f}{\partial x^2} = e^{x+y}$; $\frac{\partial^2 f}{\partial y^2} = e^{x+y}$; $\frac{\partial^3 f}{\partial x^2 \partial y} = e^{x+y}$

$\frac{\partial^3 f}{\partial x^3} = e^{x+y}$; $\frac{\partial^3 f}{\partial y^3} = e^{x+y}$; $\frac{\partial^3 f}{\partial x \partial y^2} = e^{x+y}$

At $(0,0)$ $e^{0+0} = 0$

$\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$; $\frac{\partial^2 f}{\partial x \partial y} = 0$

$\frac{\partial^2 f}{\partial x^2} = 0$; $\frac{\partial^2 f}{\partial y^2} = 0$; $\frac{\partial^3 f}{\partial x^2 \partial y} = 0$

$\frac{\partial^3 f}{\partial x^3} = 0$; $\frac{\partial^3 f}{\partial y^3} = 0$; $\frac{\partial^3 f}{\partial x \partial y^2} = 0$

Taylor's theorem:

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)^2 \cdot \frac{\partial^2}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2}{\partial x \partial y} + (y-b)^2 \frac{\partial^2}{\partial y^2}\right] f(a,b) + \frac{1}{3!}\left[(x-a)^3 \cdot \frac{\partial^3}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3}{\partial x^2 \partial y} + 3(x-a)(y-b)^2 \cdot \frac{\partial^3}{\partial x \partial y} + (y-b)^3 \cdot \frac{\partial^3}{\partial y^3}\right] f(a,b)$$

$$\Rightarrow f(x,y) = 0 + [(x-0)0 + (y-0)0] f(0,0) + \frac{1}{2} + 0 + \frac{1}{3}(0)$$

$f(x,y) =$

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Ex: Obtain Taylor's formulae for $f(x,y) = \cos(x+y)$ $n=3$ at $(0,0)$.

Sol: Given $f(x,y) = \cos(x+y)$; $n=3$; $a=0$; $b=0$

$\frac{\partial f}{\partial x} = -\sin(x+y)$; $\frac{\partial f}{\partial y} = -\sin(x+y)$; $\frac{\partial^2 f}{\partial x \partial y} = -\cos(x+y)$

$\frac{\partial^2 f}{\partial x^2} = -\cos(x+y)$; $\frac{\partial^2 f}{\partial y^2} = -\cos(x+y)$; $\frac{\partial^3 f}{\partial x^2 \partial y} = \sin(x+y)$

$\frac{\partial^3 f}{\partial x^3} = \sin(x+y)$; $\frac{\partial^3 f}{\partial y^3} = \sin(x+y)$; $\frac{\partial^3 f}{\partial x \partial y^2} = \sin(x+y)$

At point $(0,0)$

$f(0,0) = \cos(0+0) = \cos 0^\circ = 1$

$\frac{\partial f}{\partial x} = -\sin 0^\circ = 0$; $\frac{\partial f}{\partial y} = -\sin 0^\circ = 0$; $\frac{\partial^2 f}{\partial x \partial y} = -\cos 0^\circ = -1$

$\frac{\partial^2 f}{\partial x^2} = -\cos 0^\circ = -1$; $\frac{\partial^2 f}{\partial y^2} = -\cos 0^\circ = -1$

On substiting θ_x for x and θ_y for y in 3^{rd} degree partial derivatives.

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \sin(\theta_x + \theta_y)$$
$$= \sin(x+y)\theta$$

Taylor's formulae: $f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b)$
$+ \frac{1}{2!}\left[(x-a)^2 \frac{\partial^2}{\partial x^2} + 2(x-a)(y-b)\cdot\frac{\partial^2}{\partial x \partial y} + (y-b)^2 \cdot \frac{\partial^2}{\partial y^2}\right] f(a,b) +$
$\frac{1}{3!}\left[(x-a)^3 \frac{\partial^3}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3}{\partial x^3 \partial y} + 3(x-a)(y-b)^2 \frac{\partial^3}{\partial x \partial y^2} + (y-b)^3 \frac{\partial^3}{\partial y^3}\right] f(a,b) + \dots$

$\Rightarrow f(x,y) = f(0,0) + \left[x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y}\right] f(0,0) + \frac{1}{2!}\left[x^2 \frac{\partial^2}{\partial x^2} + 2xy \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2}\right] f(0,0) + \frac{1}{3!}\left[x^3 \cdot \frac{\partial^3}{\partial x^3} + 3x^2 y \frac{\partial^3}{\partial x^2 \partial y} + 3xy^2 \cdot \frac{\partial^3}{\partial x \partial y^2} + y^3 \cdot \frac{\partial^3}{\partial y^3}\right] f(\theta_x, \theta_y)$

$$1-\frac{\pi^2}{8}(x-1)^2-\frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2$$

$$\Rightarrow\quad 1+[x(0)+y(0)]+\frac{1}{2}[x^2(-1)+2xy(-1)+y^2(-1)]+$$
$$\frac{1}{6}[x^3\sin(x+y)\theta+3x^2y\sin(x+y)\theta+3xy^2\sin(x+y)\theta+y^3\sin(x+y)$$

$$\Rightarrow\quad 1-\frac{1}{2}[x^2+2xy+y^2]+\frac{1}{6}[x^3+3x^2y+3xy^2+y^3]\sin(x+y)\theta$$

$$1-\frac{1}{2}(x+y)^2+\frac{1}{6}(x+y)^3\cdot\sin(x+y)\theta$$

5. Expand $\sin(xy)$ in terms of $x-1$ and $y-\frac{\pi}{2}$ upto 2^{nd} degree terms

Sol:- Given $f(x,y)=\sin(xy)$; $a=1$; $b=\frac{\pi}{2}$

$\frac{\partial f}{\partial x}=\cos(xy)\cdot y$; $\frac{\partial f}{\partial y}=x\cdot\cos(xy)$; $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$\frac{\partial^2 f}{\partial x^2}=-y^2\sin(xy)$; $\frac{\partial^2 f}{\partial y^2}=-x^2\sin(xy)$; $\Rightarrow\frac{\partial}{\partial x}[\underset{u}{x}\cdot\underset{v}{\cos(xy)}]$

$\Rightarrow x(-\sin(xy))\cdot y+\cos(xy)$

$\Rightarrow -xy\sin(xy)+\cos(xy)$

At point $(1,\pi/2)$

$f(1,\pi/2)=\sin(1(\pi/2))\Rightarrow\sin\left(\frac{\pi}{2}\right)\Rightarrow 1$

$\frac{\partial f}{\partial x}=\frac{\pi}{2}(0)$; $\frac{\partial f}{\partial y}=0$; $\frac{\partial^2 f}{\partial x\partial y}=-\frac{\pi}{2}$

$\frac{\partial^2 f}{\partial x^2}=-\frac{\pi^2}{4}$; $\frac{\partial^2 f}{\partial y^2}=-1$

Taylor's Formulae : $f(x,y)=f(a,b)+\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a,b)$
$+\frac{1}{2!}\left[(x-a)^2\cdot\frac{\partial^2}{\partial x^2}+2(x-a)(y-b)\cdot\frac{\partial^2}{\partial x\partial y}+(y-b)^2\frac{\partial^2}{\partial y^2}\right]f(a,b)\ldots$

$$\Rightarrow f(x,y)=1+\left[(x-1)\cdot\frac{\partial}{\partial x}+\left(y-\frac{\pi}{2}\right)\frac{\partial}{\partial y}\right]f(1,\pi/2)+\frac{1}{2}\left[(x-1)\frac{\partial^2}{\partial x^2}+\right.$$
$$\left.2(x-1)(y-\pi/2)\cdot\frac{\partial^2}{\partial x\partial y}+(y-\pi/2)^2\cdot\frac{\partial^2}{\partial y^2}\right]f(1,\pi/2)$$

$$\Rightarrow f(x,y)=1+[(x-1)(0)+(y-\pi/2)(0)]+\frac{1}{2}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right)+2(x-1)(y-\pi/2)\right.$$
$$\left.-\frac{\pi}{2}+(y-\pi/2)^2(-1)\right)$$

$$\Rightarrow f(x,y)=1-\frac{\pi^2}{8}(x-1)^2-\frac{\pi}{2}(x-1)(y-\pi/2)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2$$

20m

Maximum and Minimum of a function of two Variables:-

Given function $f(x,y)$ — ①

step 1: To find the $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$

step 2: for maxima or minima $\frac{\partial f}{\partial x} = 0$ — ②

$$\frac{\partial f}{\partial y} = 0 \text{ — ③}$$

step 3: Solving eq ② & ③ we get the point $P(a,b)$

step 4: At the point $P(a,b)$ to find the value of

$$r = \frac{\partial^2 f}{\partial x^2}$$

$$s = \frac{\partial^2 f}{\partial x \partial y}$$

$$t = \frac{\partial^2 f}{\partial y^2}$$

step 5: To find the value of $rt - s^2$

Case i: if $rt - s^2 = 0$ ⇒ We cannot decided the minimum or maximum value at $P(a,b)$

Case ii: if $rt - s^2 < 0$ ⇒ $f(x,y)$ having neither minimum nor maximum at $P(a,b)$

Case iv

$$\text{if } rt-s^2>0 \begin{cases} \text{if } r>0 \text{ then } f(x,y) \text{ having minimum value of } P(a,b) \\ \text{if } r<0 \text{ then } f(x,y) \text{ having maximum value of } P(a,b) \end{cases}$$

Minimum value or maximum value are equal to $f(a,b)$

v.imp *

Show that minimum value of

$$U = xy + \frac{a^3}{x} + \frac{a^3}{y} \text{ is } 3a^2$$

Given,

$$U = xy + \frac{a^3}{x} + \frac{a^3}{y} \quad \text{--- (1)}$$

$$\frac{\partial U}{\partial x} = y(1) + a^3\left(\frac{-1}{x^2}\right) + 0$$

$$\frac{\partial U}{\partial x} = y - \frac{a^3}{x^2}$$

$$\frac{\partial^2 U}{\partial x^2} = 0 - a^3 \frac{\partial}{\partial x}\left(\frac{1}{x^2}\right)$$

$$= -a^3 \frac{\partial}{\partial x}(x^{-2})$$

$$= -a^3 (-2) x^{-2-1}$$

$$= a^3 2 x^{-3}$$

$$\frac{\partial^2 U}{\partial x^2} = 2\frac{a^3}{x^3}$$

$$\frac{\partial U}{\partial y} = x(1) + 0 + a^3\left(\frac{-1}{y^2}\right)$$

$$\frac{\partial U}{\partial y} = x - \frac{a^3}{y^2}$$

$$\frac{\partial^2 U}{\partial y^2} = 0 - a^3 \cdot \frac{\partial}{\partial y}(y^{-2})$$

$$= -a^3 (-2) y^{-2-1}$$

$$= a^3 2 y^{-3}$$

$$\frac{\partial^2 U}{\partial y^2} = 2\frac{a^3}{y^3}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(x - \frac{a^3}{y^2}\right)$$

$$= 1 - 0$$

$$\frac{\partial^2 u}{\partial x \partial y} = 1$$

for maxima ~~or~~ minima $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

$y - \frac{a^3}{x^2} = 0$ $\qquad$ $x - \frac{a^3}{y^2} = 0$

$y = \frac{a^3}{x^2}$ $\qquad$ $x = \frac{a^3}{y^2}$

$x^2 y = a^3$ — (2) $\qquad$ $xy^2 = a^3$ — (3)

equating (2) & (3)

$$x^2 y = xy^2$$

$$x^2 y - xy^2 = 0$$

$$xy(x - y) = 0$$

$xy = 0$ $\qquad$ $x - y = 0$

$$\boxed{x = y}$$

$x = y$ sub in eq (2)

$$x^2 \cdot x = a^3$$

$$x^3 = a^3$$

$\boxed{x = a}$ $\quad$ $\boxed{y = a}$

Point of $P(x,y)$ is $P(a,a)$

$$r = \frac{\partial^2 u}{\partial x^2} = 2\frac{a^3}{x^3} = 2\frac{a^3}{a^3} = 2$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2 \cdot \frac{a^3}{y^3} = 2 \cdot \frac{a^3}{a^3} = 2$$

$$rt - s^2 \Rightarrow 2(2) - 1^2 \Rightarrow 4-1$$
$$= 3$$

$rt - s^2 = 3$

$rt - s^2 > 0$ and $r > 0$

$\therefore$ u has minimum value at $P(a,a)$

Minimum value $= u(a,a)$

$$= a \cdot a + \frac{a^3}{a} + \frac{a^3}{a}$$

$$= a^2 + \frac{a^3}{a} + \frac{a^3}{a}$$

$$= a^2 + a^2 + a^2$$

$$= 3a^2$$

Hence proved

find the minimum or maximum of $f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$

sol $(3,1)$

min value $= 9$

find the minimum or maximum value of

$$f(x,y) = xy[6-x-y]$$

Given,

$$f(x,y) = xy[6-x-y] \quad -(1)$$

$$f(x,y) = 6xy - x^2y - xy^2$$

$$\frac{\partial f}{\partial x} = 6y(1) - y(2x) - y^2(1)$$

$$\frac{\partial f}{\partial x} = 6y - 2xy - y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 0 - 2y(1) - 0$$

$$\frac{\partial^2 f}{\partial x^2} = -2y$$

$$\frac{\partial f}{\partial y} = 6x(1) - x^2(1) - x(2y)$$

$$\frac{\partial f}{\partial y} = 6x - x^2 - 2xy$$

$$\frac{\partial^2 f}{\partial y^2} = 0 - 0 - 2x(1)$$

$$\frac{\partial^2 f}{\partial y^2} = -2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(6x - x^2 - 2xy)$$

$$= 6(1) - 2x - 2y(1)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6 - 2x - 2y$$

for maxima or minima $\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0$

$6y - 2xy - y^2 = 0$

$y[6 - 2x - y] = 0$

$y = 0 \quad 6 - 2x - y = 0$

$-2x - y + 6 = 0$

$2x + y - 6 = 0$ — (2)

$6x - x^2 - 2xy = 0$

$x[6 - x - 2y] = 0$

$x = 0 \quad 6 - x - 2y = 0$

$-x - 2y + 6 = 0$

$x + 2y - 6 = 0$ — (3)

solving (2) & (3)

(2) × 2 $\quad 4x + \cancel{2y} - 12 = 0$

(3) × 1 $\quad x + \cancel{2y} - 6 = 0$

$\quad\;\; - \quad - \quad +$

$3x - 6 = 0$

$3x = 6$

$\boxed{x = 2}$

(2,2)

maxi = 8

(0,0)

neither max & min

Sub. x in eq (2)

$2(2) + y - 6 = 0$

$4 + y - 6 = 0$

$\boxed{y = 2}$

$\therefore$ Point $P(2,2)$ and $P(0,0)$

$r = \frac{\partial^2 f}{\partial x^2} = -2y = -2(2) = -4$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6 - 2x - 2y = 6 - 2(2) - 2(2) = -2$

$4 \quad -8 \quad 4$

$t = \frac{\partial^2 f}{\partial y^2} = -2x = -2(2) = -4$

$rt - s^2 = (-4)(-4) - (2)^2 = 16 - 4$

$\Rightarrow 12$

$rt - s^2 = 12$

$rt - s^2 > 0$ and $r < 0$

$\therefore$ f has a maximum value at $P(2,2)$ & $P(0,0)$

case I:
maximum value $\Rightarrow f(2,2)$

$\Rightarrow 6xy - x^2y - xy^2$

$= 6(2)(2) - (2)^2(2) - 2(2)^2$

$= 6(4) - (4)(2) - 2(4)$

$= 24 - 8 - 8$

$= 24 - 16$

$= 8$

case II $P(0,0)$

$f(0,0)$

$6(0)(0) - 0 - 0$

$\Rightarrow 0$

$\therefore$ Nether maximum nor minimum.

(2) find the max & mini of $f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$

Given, $f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$

$\frac{\partial f}{\partial x} = y(1) + 9\left(\frac{-1}{x^2}\right) + 0$

$\frac{\partial f}{\partial x} = y - \frac{9}{x^2}$

$\frac{\partial f}{\partial y} = x(1) + 0 + 3\left(\frac{-1}{y^2}\right)$

$\frac{\partial f}{\partial y} = x - \frac{3}{y^2}$

$\frac{\partial^2 f}{\partial x^2} = 0 - 9\frac{\partial}{\partial x}(x^{-2})$

$= -9(-2)x^{-2-1}$

$= 18x^{-3}$

$\frac{\partial^2 f}{\partial x^2} = \frac{18}{x^3}$

$\frac{\partial^2 f}{\partial y^2} = 0 - 3\frac{\partial}{\partial y}(y^{-2})$

$= -3(-2)y^{-2-1}$

$= 6y^{-3}$

$\frac{\partial^2 f}{\partial y^2} = \frac{6}{y^3}$

$\frac{\partial f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left(x - \frac{3}{y^2}\right)$

$= 1 - 0$

$\frac{\partial f}{\partial x \partial y} = 1$

for maxima & minima $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$y - \frac{9}{x^2} = 0$

$y = \frac{9}{x^2}$

$y = \frac{9}{\left(\frac{3}{y^2}\right)^2}$

$= 9 \cdot \left(\frac{y^2}{3}\right)^2$

$y = \cancel{9}\,\frac{y^4}{\cancel{9}}$

$y = y^4$

$y - y^4 = 0$

$y(1-y^3) = 0$

$y = 0 \quad 1 - y^3 = 0$

$1 = y^3$

$y^3 = 1$

$\boxed{y = 1}$

$x - \frac{3}{y^2} = 0$

$x = \frac{3}{y^2}$

$x = \frac{3}{(1)^2}$

$x = \frac{3}{1}$

$\boxed{x = 3}$

$\therefore$ Point $P(x, y) = P(3, 1)$

$r = \frac{\partial^2 U}{\partial x^2} = \frac{18}{x^3} = \frac{18}{(3)^3} = \frac{18}{27} = 0.6$

$s = \frac{\partial^2 U}{\partial x \partial y} = 1$

$t = \frac{\partial^2 U}{\partial y^2} = \frac{6}{y^3} = \frac{6}{(1)^3} = 6$

$rt - s^2 = (0.6)(6) - (1)^2 = 3.6 - 1 \Rightarrow 2.6$

$rt - s^2 = 2.6$

$rt - s^2 > 0$ & $r > 0$

$\therefore f$ is a minimum at $P(3,1)$

Minimum value $= f(3,1)$

$$= xy + \frac{9}{x} + \frac{3}{y}$$

$$= (3)(1) + \frac{\cancel{9}^{3}}{\cancel{3}} + \frac{3}{1}$$

$$= 3 + 3 + 3$$

$$= 9$$

Example: Discuss the maximum and minimum values of U given by $U = x^3y^2(1-x-y)$

Sol Given that, $U = x^3y^2(1-x-y)$

$$U = x^3y^2 - x^4y^2 - x^3y^3$$

$$\frac{\partial U}{\partial x} = y^2(3x^2) - y^2(4x^3) - y^3(3x^2)$$

$$\frac{\partial U}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial^2 U}{\partial x^2} = 3y^2(2x) - 4y^2(3x^2) - 3y^3(2x)$$

$$\frac{\partial^2 U}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$\frac{\partial U}{\partial y} = x^3(2y) - x^4(2y) - x^3(3y^2)$$

$$\frac{\partial U}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

$$\frac{\partial^2 U}{\partial y^2} = 2x^3(1) - 2x^4(1) - 3x^3(2y)$$

$$\frac{\partial^2 U}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(2x^3y - 2x^4y - 3x^3y^2\right)$$

$$= 2y(3x^2) - 2y(4x^3) - 3y^2(3x^2)$$

$$\frac{\partial^2 U}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

for maxima or minima $\frac{\partial U}{\partial x} = 0$ $\frac{\partial U}{\partial y} = 0$

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(3 - 4x - 3y) = 0$$

$x^2y^2 = 0$ (or) $3 - 4x - 3y = 0$

$x = 0$ or $y = 0$

$$-4x - 3y + 3 = 0$$

$$4x + 3y - 3 = 0 \quad -(2)$$

$$2x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y[2 - 2x - 3y] = 0$$

$x^3y = 0$

$x = 0$ (or) $y = 0$

$$2 - 2x - 3y = 0$$

$$-2x - 3y + 2 = 0$$

$$2x + 3y - 2 = 0 \quad -(3)$$

solving eq (2) & (3)

$$4x + 3y - 3 = 0$$
$$2x + 3y - 2 = 0$$
$$- \quad - \quad +$$
$$2x - 1 = 0$$
$$2x = 1$$
$$\boxed{x = 1/2}$$

sub in eq (2)

$$4(1/2) + 3(y) - 3 = 0$$
$$\cancel{2 + \frac{3}{2} - 3}$$
$$2 + 3y - 3 = 0$$
$$3y - 1 = 0$$
$$\boxed{y = 1/3}$$

$\therefore$ Point $P(x, y) = (1/2, 1/3)$

$$r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$
$$= 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3$$
$$= 6\left(\frac{1}{2}\right)\left(\frac{1}{9}\right) - 12\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) - 6\left(\frac{1}{2}\right)\left(\frac{1}{27}\right)$$
$$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9}$$
$$= \frac{3 - 3 - 1}{9} = \frac{-1}{9}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$
$$= 6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) - 8\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right) - 9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2$$
$$= 6\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) - 8\left(\frac{1}{8}\right)\left(\frac{1}{3}\right) - 9\left(\frac{1}{4}\right)\left(\frac{1}{9}\right)$$
$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$$
$$\Rightarrow \frac{6 - 4 - 3}{12} = \frac{-1}{12}$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$
$$= 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)$$
$$= 2\left(\frac{1}{8}\right) - 2\left(\frac{1}{16}\right) - 6\left(\frac{1}{8}\right)\left(\frac{1}{3}\right)$$
$$= \frac{1}{4} - \frac{1}{8} - \frac{1}{4}$$
$$\Rightarrow \frac{2 - 1 - 2}{8} = \frac{-1}{8}$$

$$rt - s^2 \Rightarrow \left(\frac{-1}{9}\right)\left(\frac{-1}{8}\right) - \left(\frac{-1}{12}\right)^2$$
$$\Rightarrow \frac{1}{72} - \frac{1}{144}$$
$$\Rightarrow \frac{2 - 1}{144} = \frac{1}{144}$$

$rt - s^2 > 0$

$r < 0$

$\therefore$ maximum value at $P\left(\frac{1}{2}, \frac{1}{3}\right)$

$Q(0,0)$

$rt - s^2 = 0$

$r = 0$

neither maximum nor mini

mini value = $x^3 y^2(1-x-y)$

$$= \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{8} \times \frac{1}{9} \times \frac{1}{6}$$

$2\times2\times2\times2$

$$\boxed{U = \frac{1}{432}}$$

NOTE:

$\cos(180-\theta) = -\cos\theta$

$\sin(180-\theta) = \sin\theta$

$\cos 180 = -1$

$\sin 180 = 0$

$\sin A \cos B + \cos A \sin B = \sin(A+B)$

$\cos A \cos B - \sin A \sin B = \cos(A+B)$

* In a plane triangle find the max value of

$$U = \cos A \cos B \cos C$$

Given $A+B+C=180$

$$A+B = 180-C$$

$$U = \cos A \cos B \cos C$$

$$U = \cos A \cos B \cos(180-(A+B))$$

$$U = -\cos A \cos B \cos(A+B)$$

$$\frac{\partial U}{\partial A} = -\cos B\left[\cos A \cdot \frac{\partial}{\partial A}\cos(A+B) + \cos(A+B)\frac{\partial}{\partial A}(\cos A)\right]$$

$$= -\cos B\left[-\sin(A+B)\cos A + \cos(A+B)(-\sin A)\right]$$

$$= \cos B\left[\sin(A+B)\cos A + \cos(A+B)\sin A\right]$$

$$= \cos B \sin((A+B)+A)$$

$$\frac{\partial u}{\partial A} = \cos B \sin(2A+B)$$

$$\frac{\partial^2 u}{\partial A^2} = \cos B \frac{\partial}{\partial A}(\sin(2A+B))$$

$$= \cos B \cos(2A+B)(2)$$

$$r = \frac{\partial^2 u}{\partial A^2} = 2\cos B \cos(2A+B)$$

$$\frac{\partial u}{\partial B} = -\cos A\left[\cos B \cdot \frac{\partial}{\partial B}\cos(A+B) + \cos(A+B)\cdot\frac{\partial}{\partial B}(\cos B)\right]$$

$$= -\cos A\left[\cos B(-\sin(A+B)) + \cos(A+B)(-\sin B)\right]$$

$$= \cos A\left[\sin(A+B)\cos B + \cos(A+B)\sin B\right]$$

$$= \cos A\left[\sin((A+B)+B)\right]$$

$$\frac{\partial u}{\partial B} = \cos A \sin(A+2B)$$

$$\frac{\partial^2 u}{\partial B^2} = \cos A \cos(A+2B)(2)$$

$$t = \frac{\partial^2 u}{\partial B^2} = 2\cos A \cos(A+2B)$$

$$\frac{\partial^2 u}{\partial A \partial B} = \frac{\partial}{\partial A}\left(\frac{\partial u}{\partial B}\right)$$

$$= \frac{\partial}{\partial A}(\cos A \cdot \sin(A+2B))$$

$= \cos A \cos(A+2B) + \sin(A+2B)(-\sin A)$

$= \cos A \cos(A+2B) - \sin A \sin(A+2B)$

$= \cos(A+(A+2B)]$

$= \cos(2A+2B)$

for max & min

$\frac{\partial u}{\partial A} = 0$

$\cos B \sin(2A+B) = 0$

$\cos B = 0$ | $\sin(2A+B) = 0$

$2A+B = 180$ — (1)

$\frac{\partial u}{\partial B} = 0$

$\cos A \sin(A+2B) = 0$

$\cos A = 0$ | $\sin(A+2B) = 0$

$A+2B = 180$ — (2)

solving (1) & (2)

$$\begin{aligned} eq(1) \times 2 \Rightarrow 4A + \cancel{2B} &= 360 \\ A + \cancel{2B} &= 180 \\ - \quad - \quad & \quad - \\ \hline 3A &= 180 \end{aligned}$$

$\boxed{A = 60}$

$A=60$ sub in eq (1)

$2(60) + B = 180$

$B = 60$

$C = 60$

$$r = \frac{\partial^2 U}{\partial A^2} = 2\cos B \cos(2A+B)$$

$$= 2\cos 60 \cos(2(60)+60)$$

$$= 2\cos 60 \cos(120+60)$$

$$= 2\cos 60 \cos 180$$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}} \cdot (-1)$$

$$\boxed{r = -1}$$

$$s = \frac{\partial^2 U}{\partial A \partial B} = \cos(2A+2B)$$

$$= \cos(240)$$

$$= \cos(270-30)$$

$$= -\sin 30$$

$$\boxed{s = -1/2}$$

$$t = \frac{\partial^2 U}{\partial B^2} = 2\cos A \cos(A+2B)$$

$$= 2\cos 60 \cos(60+2(60))$$

$$= 2\cos 60 \cos 180$$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}} \cdot (-1)$$

$$\boxed{t = -1}$$

$$rt - s^2 = (-1)(-1) - \left(\frac{-1}{2}\right)^2$$

$$= 1 - \frac{1}{4}$$

$$rt - s^2 = \frac{3}{4}$$

$rt - s^2 > 0$ & $r < 0$

U has max value at $A = B = C = 60^\circ$

Maximum & minimum values.

Q) Discuss max or min $U = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$U = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$\cancel{\frac{\partial U}{\partial x} = 4x^3 - 4y^3 x}$$

$$\frac{\partial U}{\partial x} = 4x^3 + 0 - 2(2x) + 4y(1) - 0$$

$$\frac{\partial U}{\partial x} = 4x^3 - 4x + 4y \quad , \quad \frac{\partial^2 U}{\partial x^2} = 12x^2 - 4$$

$$\frac{\partial U}{\partial y} = 0 + 4y^3 + 4x - 4y$$

$$\frac{\partial U}{\partial y} = 4y^3 + 4x - 4y \quad , \quad \frac{\partial^2 U}{\partial y^2} = 12y^2 - 4$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[4y^3 + 4x - 4y\right] \Rightarrow 0 + 4(1) - 0$$

$$= 4$$

for max & min value

$$\frac{\partial u}{\partial x} = 0 \qquad\qquad \frac{\partial u}{\partial y} = 0$$

$$4x^3 - 4x + 4y = 0 \qquad\qquad 4y^3 + 4x - 4y = 0$$

$$x^3 - x + y = 0 \quad \text{—(1)} \qquad\qquad y^3 + x - y = 0 \quad \text{—(2)}$$

solve (1) & (2)

$$x^3 - \cancel{x} + \cancel{y} = 0$$

$$y^3 + \cancel{x} - \cancel{y} = 0$$

$$x^3 + y^3 = 0$$

$$\boxed{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$

$$(x+y)(x^2 - xy + y^2) = 0$$

$x + y = 0$ | $x^2 - xy + y^2 = 0$

$y = -x$ | $\rightarrow$ complex roots

sub in (1)

$$x^3 - x + y = 0$$

$$x^3 - x - x = 0$$

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0$$

$x = 0$ | $x^2 = 2$

| $x = \pm\sqrt{2}$

when $x=0$	$x=-\sqrt{2}$	$x=\sqrt{2}$
$y=-x$	$x=-x$	$y=-x$
$y=0$	$y=\sqrt{2}$	$y=-\sqrt{2}$

points $P(0,0)$ $Q(\sqrt{2},-\sqrt{2})$ $R(-\sqrt{2},\sqrt{2})$

Case i: $P(0,0)$

$r=\dfrac{\partial^2 f}{\partial x^2}=12x^2-4$ $\quad s=\dfrac{\partial^2 u}{\partial x\partial y}$ $\quad t=\dfrac{\partial^2 u}{\partial y^2}=12y^2-4$

$\boxed{r=-4}$ $\quad \boxed{s=4}$ $\quad \boxed{t=-4}$

$rt-s^2=(-4)(-4)-(4)^2$

$=0$

We can't decide

Case ii: $Q(\sqrt{2},-\sqrt{2})$

$r=12(\sqrt{2})^2-4$, $\boxed{s=4}$, $t=12(-\sqrt{2})^2-4$

$=24-4$ $\quad \boxed{t=-20}$

$\boxed{r=20}$

$rt-s^2=(20)(-20)-(4)^2 \Rightarrow -400-16.$

Case iii: at $R(-\sqrt{2},\sqrt{2})$

$rt-s^2 = (20)(20)-16$

$= 384$

mini value $U = x^4+y^4-2x^2+4xy-2y^2$

$= (\sqrt{2})^4-(-\sqrt{2})^4-2(\sqrt{2})^2+4(\sqrt{2})(-\sqrt{2})-2(\sqrt{2})^2$

$= ((\sqrt{2})^2)^2-((-\sqrt{2})^2)^2-4-4(\sqrt{2})^2-4$

$= 4+4-4-8-4$

$= -8$

for both min value is -8.

(5) $f(x,y) = x^3+y^3-3x-12y+2=0$

Given

$f(x,y) = x^3+y^3-3x-12y+2=0$

$\frac{\partial f}{\partial x} = 3x^2+0-3(1)+0+0$

$\frac{\partial f}{\partial x} = 3x^2-3$

$\frac{\partial^2 f}{\partial x^2} = 6x$

$\frac{\partial f}{\partial y} = 0+3y^2-0-12(1)$

$\frac{\partial f}{\partial y} = 3y^2-12$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$\frac{\partial^2 f}{\partial y^2} = 6y$

$= \frac{\partial}{\partial x}(3y^2-12)$

$= 0$

for min or max

$\frac{\partial f}{\partial x} = 0$

$3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$

$\frac{\partial f}{\partial y} = 0$

$3y^2 - 12 = 0$

$y^2 = 4$

$y = \pm 2$

$P(1,2)$, $Q(1,-2)$, $R(-1,2)$, $S(-1,-2)$

<u>Case i:</u> at point $P(1,2)$

$r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

$= 6(x)$ $\boxed{s = 0}$ $= 6y$

$= 6(1)$ $\boxed{t = 12}$

$\boxed{r = 6}$

$rt - s^2 = 6(12) = 72 > 0$ & $r > 0$ and f has min value

min value $= P(1,2)$

$f = (1)^3 + (2)^3 - 3(1) - 12(2) + 2$

$= 1 + 8 - 3 - 24 + 2$

$= 11 - 27$

$\boxed{f = -16}$

Case ii: $Q(1,-2)$

$r = 6(x)$, $s = 0$, $t = 6y$

$\boxed{r = 6}$ $\boxed{t = -12}$

$rt - s^2 = -72 \Rightarrow rt - s^2 < 0$

f has neither max or mini

Case iii: $R(-1,2)$

$r = -6,\ t = 12,\ s = 0$

$rt - s^2 \Rightarrow (-6)(12) - 0$

$= -72 < 0$

f has neither mini or max.

case iv: $S(-1,-2)$

$r = -6,\ t = -12,\ s = 0$

$rt - s^2 = (-6)(-12) - 0$

$= 72$

$rt - s^2 > 0$ & r is < 0 then f has max value

$f(-1,2) = x^3 + y^3 - 3x - 12y + 2$

$= -1 - 8 + 3 + 24 + 2$

$= 20.$

Q) discuss mini or max values $f(x,y) = x^4+2x^2y-x^2+3y^2$

Given that,

$$f(x,y) = x^4+2x^2y-x^2+3y^2$$

$$\frac{\partial f}{\partial x} = 4x^3+4xy-2x \quad , \quad \frac{\partial f}{\partial y} = 2x^2+6y$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2+4y-2 \qquad \frac{\partial^2 f}{\partial y^2} = 6$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(2x^2+6y\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4x$$

for mini or max

$$\frac{\partial f}{\partial x} = 0 \quad , \quad \frac{\partial f}{\partial y} = 0$$

$$4x^3+4xy-2x=0 \quad , \quad 2x^2+6y=0$$

$$x(4x^2+4y-2)=0 \quad -(1) \qquad 2x^2+6y=0 \quad -(2)$$

$x=0 \mid 4x^2+4y-2=0$

solving eq (1) & (2)

eq ① $\cancel{4x^2}+4y-2=0$

eq ② × 2 $\cancel{4x^2}+12y=0$

$-8y-2=0$

$-8y=2$

$\boxed{y=-1/4}$

sub $y=-1/4$ in eq ①

$4x^2+4\left(\frac{-1}{4}\right)=2$

$4x^2=2+1$

$4x^2=3$

$x=\pm\sqrt{3/4}$

$P\left(\frac{\sqrt{3}}{2},\frac{-1}{4}\right)$

$r=12\left(\frac{\sqrt{3}}{2}\right)^2+4\left(\frac{-1}{4}\right)-2$

$=\cancel{12}^{3}\left(\frac{3}{\cancel{4}}\right)+4\left(\frac{-1}{4}\right)-2$

$=9-1-2$

$\boxed{r=6}$

$s=4x$

$=\cancel{4}^{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)$

$\boxed{s=2\sqrt{3}}$

$\boxed{t=6}$

$rt-s^2=(6)(6)-(2\sqrt{3})^2$

$=36-4(3)\Rightarrow 24>0$

∴ f has mini value.

$$\text{mini value} = \left(\frac{\sqrt{3}}{2}\right)^4 + 2\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{-1}{4}\right) - \left(\frac{\sqrt{3}}{2}\right)^2 + 3\left(\frac{-1}{4}\right)^2$$

$$= \frac{9}{16} + 2\left(\frac{3}{4}\right)\left(\frac{-1}{4}\right) - \frac{3}{4} + 3\left(\frac{1}{16}\right)$$

$$= -3/8$$

$f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$

Given,

$$f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$$

$$\frac{\partial f}{\partial x} = 2x - y + 3 \quad , \quad \frac{\partial f}{\partial y} = -x + 2y - 2$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-x + 2y - 2\right)$$

$$= -1.$$

for mini or max

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$2x - y + 3 = 0 \qquad -x + 2y - 2 = 0$$

$$2x - y = -3 \quad -(1) \qquad x - 2y = -2 \quad -(2)$$

solving. eq(1) & (2)

eq(1) $\quad \cancel{2x} - y = -3$

eq(2) ×2 $\quad \cancel{2x} - 4y = -4$

$$\overline{\quad 3y = 1 \quad}$$

$$\boxed{y = 1/3}$$

sub in eq(1)

$2x - \frac{1}{3} = -3$

$6x - 1 = -9$

$6x = -9 + 1$

$6x = -8$

$x = \frac{-\cancel{8}\,4}{\cancel{6}\,3}$

$$\boxed{x = \frac{-4}{3}}$$

Point $P\left(-\frac{4}{3}, \frac{1}{3}\right)$

$r = \frac{\partial^2 f}{\partial x^2} = 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = -1$

$t = \frac{\partial^2 f}{\partial y^2} = 2$

$rt - s^2 = (2)(2) - (-1)^2$

$rt-s^2 = 3$

$rt-s^2>0$ & $r>0$ & f has mini value

$$f\left(\frac{-4}{3}, \frac{1}{3}\right) = \left(\frac{-4}{3}\right)^2 - \left(\frac{-4}{3}\right)\left(\frac{1}{3}\right) + 3\left(\frac{-4}{3}\right) - 2\left(\frac{1}{3}\right) + 1 + \left(\frac{1}{3}\right)^2$$

$$= \frac{16}{9} + \frac{4}{9} - 4 - \frac{2}{3} + 1 + \frac{1}{9}$$

$$= \frac{4}{3} //$$

Minimum or Maximum value of the functions of 3 variables

lagranges Method of undetermined multipliers.

Given f^n $f(x,y,z)$ — ①

Given condition $\phi(x,y,z) = 0$ — ②

lagranges function as

$$F(x,y,z) = f(x,y,z) + \phi(x,y,z)$$

To find $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$

① find the minimum value of $x^2+y^2+z^2$ given condition is $x+y+z = 3a$

$$f(x,y,z) = x^2+y^2+z^2 \text{ — ①}$$

$$\phi(x,y,z) = x+y+z-3a \text{ — ②}$$

lagrangs theorem

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$$

$$F(x,y,z) = x^2 + y^2 + z^2 + \lambda(x+y+z-3a)$$

$$\frac{\partial F}{\partial x} = 2x + 0 + 0 + \lambda(1+0+0-0)$$

$$\frac{\partial F}{\partial x} = 2x + \lambda$$

$$\frac{\partial F}{\partial y} = 0 + 2y + 0 + \lambda(0+1+0-0)$$

$$\frac{\partial F}{\partial y} = 2y + \lambda$$

$$\frac{\partial F}{\partial z} = 2z + \lambda$$

for max or mini

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial z} = 0$$

$$2x + \lambda = 0 \qquad 2y + \lambda = 0 \qquad 2z + \lambda = 0$$

$$\lambda = -2x \quad — (3) \qquad \lambda = -2y \quad — (4) \qquad \lambda = -2z \quad — (5)$$

from (3), (4), (5)

$$-2x = -2y = -2z$$

$$x = y = z$$

sub in eq (2)

$3x = 3a$

$\boxed{x = a}$

minimum value $= a^2 + a^2 + a^2$

$= 3a^2$

(3) Find the minimum value of ~~y~~ $x+y+z$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$f(x, y, z) = x + y + z$ — (1)

$\phi(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$ — (2)

lagranges function

$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

$F(x, y, z) = x + y + z + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$

$\frac{\partial F}{\partial x} = 1 - \frac{\lambda}{x^2}$, $\frac{\partial F}{\partial y} = 1 - \frac{\lambda}{y^2}$, $\frac{\partial F}{\partial z} = 1 - \frac{\lambda}{z^2}$

for max or mini

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$1 - \frac{\lambda}{x^2} = 0$	$y^2 = \lambda$	$z^2 = \lambda$
$\frac{x^2 - \lambda}{x^2} = 0$		
$x^2 - \lambda = 0$ $\boxed{x^2 = \lambda}$		

$$x^2 = y^2 = z^2 = \lambda$$

$$x = y = z$$

sub in eq (2)

$$\frac{1}{x} + \frac{1}{x} + \frac{1}{x} = 1$$

$$\frac{3}{x} = 1$$

$$\boxed{x = 3}$$

sub in eq (1)

$$= x + y + z$$

$$= 3 + 3 + 3$$

$$= 9$$

(*) find the mini value of $x^2 + y^2 + z^2$ given condition

$$ax + by + cz = p$$

$$f(x, y, z) = x^2 + y^2 + z^2 \quad -(1)$$

$$\phi(x, y, z) = ax + by + cz - p \quad -(2)$$

by lagrange's theorem

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$$

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda(ax + by + cz)$$

$$\frac{\partial F}{\partial x} = 2x + \lambda a \qquad \frac{\partial F}{\partial y} = 2y + \lambda b \qquad \frac{\partial F}{\partial z} = 2z + \lambda c$$

$$\frac{\partial^2 F}{\partial x^2} = 2 \qquad \frac{\partial^2 F}{\partial y^2} = 2 \qquad \frac{\partial^2 F}{\partial z} = 2$$

for max or mini

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial z} = 0$$

$2x+\lambda a=0$ | $2y+\lambda b=0$ | $2z+\lambda c=0$

$x=\frac{-\lambda a}{2}$ | $y=\frac{-\lambda b}{2}$ | $z=\frac{-\lambda c}{2}$

sub x,y,z values in eq (2)

$$ax+by+cz=p$$

$$a\left(\frac{-\lambda a}{2}\right)+b\left(\frac{-\lambda b}{2}\right)+c\left(\frac{-\lambda c}{2}\right)=p$$

$$\lambda a^2+\lambda b^2+\lambda c^2=-2p$$

$$\boxed{\lambda=\frac{-2p}{a^2+b^2+c^2}}$$

$$x=\frac{-a}{2}\left(\frac{-2p}{a^2+b^2+c^2}\right)=\frac{ap}{a^2+b^2+c^2}$$

$$y=\frac{-b}{2}\left(\frac{-2p}{a^2+b^2+c^2}\right)=\frac{bp}{a^2+b^2+c^2}$$

$$z=\frac{-c}{2}\left(\frac{-2p}{a^2+b^2+c^2}\right)=\frac{cp}{a^2+b^2+c^2}$$

sub in eq (1)

$$f=x^2+y^2+z^2=\left(\frac{ap}{a^2+b^2+c^2}\right)^2+\left(\frac{bp}{a^2+b^2+c^2}\right)^2+\left(\frac{cp}{a^2+b^2+c^2}\right)^2$$

$$=\frac{a^2p^2+b^2p^2+c^2p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2\cancel{(a^2+b^2+c^2)}}{(a^2+b^2+c^2)^2}$$

$$f = \frac{p^2}{a^2+b^2+c^2} \;//$$

7) find the maximum & minimum value of

$U = a^2x^2+b^2y^2+c^2z^2$ where $x^2+y^2+z^2=1$ &

$lx+my+nz=0$

Given, $U = a^2x^2+b^2y^2+c^2z^2$ — ①

$x^2+y^2+z^2-1=0$ — ②

$lx+my+nz=0$ — ③

for lagrange's theorem

$$f = U + \lambda_1\phi_1 + \lambda_2\phi_2$$

$$f = (a^2x^2+b^2y^2+c^2z^2) + \lambda_1(x^2+y^2+z^2-1) + \lambda_2(lx+my+nz)$$

$$\frac{\partial F}{\partial x} = 2a^2x + \lambda_1(2x) + \lambda_2(l)$$

$$\frac{\partial F}{\partial y} = 2b^2y + 2\lambda_1 y + \lambda_2(m)$$

$$\frac{\partial F}{\partial z} = 2c^2z + 2\lambda_1 z + \lambda_2(n)$$

for max or mini

$$\frac{\partial f}{\partial x}=0 \quad , \quad \frac{\partial f}{\partial y}=0 \quad , \quad \frac{\partial f}{\partial z}=0$$

$$2a^2x+2\lambda_1 x+\lambda_2 l=0 \quad , \quad 2b^2y+2\lambda_1 y+\lambda_2 m=0 \quad , \quad 2c^2z+2\lambda_1 z+\lambda_2 n=0$$

— (4) — (5) — (6)

$x \cdot eq(4) + y \cdot eq(5) + z\, eq(6)$

$$2(a^2x^2+b^2y^2+c^2z^2)+2\lambda_1(x^2+y^2+z^2)+\lambda_2(lx+my+nz)=0$$

$$2U+2\lambda_1(1)+\lambda_2(0)=0$$

$$2U+2\lambda_1=0$$

$$2\lambda_1=-2U$$

$$\boxed{\lambda_1=-U}$$

λ_1 value sub in eq (4)

$$2a^2x+2\lambda_1 x+l\lambda_2=0$$

$$2a^2x+2(-U)x+l\lambda_2=0$$

$$2a^2x-2Ux=-l\lambda_2$$

$$2x(a^2-U)=-l\lambda_2$$

$$\boxed{x=\frac{-l\lambda_2}{2(a^2-U)}}$$

λ_1 value in eq (5)

$$2b^2y + 2\lambda_1 y + \lambda_2 m = 0$$

$$2b^2y - 2uy = -\lambda_2 m$$

$$2y(b^2 - u) = -\lambda_2 m$$

$$\boxed{y = \frac{-\lambda_2 m}{2(b^2-u)}}$$

λ_1 in eq (6)

$$2c^2z + 2\lambda_1 z + \lambda_2 n = 0$$

$$2c^2z - 2uz = -\lambda_2 n$$

$$\boxed{z = \frac{-\lambda_2 n}{2(c^2-u)}}$$

x, y, z value sub in eq (3)

$$lx + my + nz = 0$$

$$l\left(\frac{-l\lambda_2}{2(a^2-u)}\right) + m\left(\frac{-\lambda_2 m}{2(b^2-u)}\right) + n\left(\frac{-\lambda_2 n}{2(c^2-u)}\right) = 0$$

$$\frac{-\lambda_2}{2}\left[\frac{l^2}{a^2-u} + \frac{m^2}{b^2-u} + \frac{n^2}{c^2-u}\right] = 0$$

$$\frac{l^2}{a^2-u} + \frac{m^2}{b^2-u} + \frac{n^2}{c^2-u} = 0 \;//$$

(4) find the mini & max value of $U = x^2+y^2+z^2$. where

$ax^2+by^2+cz^2=1$ & $lx+my+nz=0$

Given that, $U = x^2+y^2+z^2$ — ①

$ax^2+by^2+cz^2=1$ — ②

$lx+my+nz=0$ — ③

Let, $\phi_1 = ax^2+by^2+cz^2-1$

$\phi_2 = lx+my+nz$

By lagrang's therem

$f = U+\lambda_1\phi_1+\lambda_2\phi_2$

$f = (x^2+y^2+z^2)+\lambda_1(ax^2+by^2+cz^2-1)+\lambda_2(lx+my+nz)$

$\frac{\partial f}{\partial x} = 2x+2a\lambda_1 x+\lambda_2 l$

$\frac{\partial f}{\partial y} = 2y+2by\lambda_1+\lambda_2 m$

$\frac{\partial f}{\partial z} = 2z+2cz\lambda_1+\lambda_2 n$

for maxi or mini

$\frac{\partial f}{\partial x}=0$	$\frac{\partial f}{\partial y}=0$	$\frac{\partial f}{\partial z}=0$
$2x+2a\lambda_1 x+\lambda_2 l=0$ — ④	$2y+2by\lambda_1+\lambda_2 m=0$ — ⑤	$2z+2cz\lambda_1+\lambda_2 n=0$ — ⑥

$x \cdot eq(4) + y \cdot eq(5) + z \cdot eq(6)$

$$2(x^2+y^2+z^2) + 2\lambda_1(ax^2+by^2+cz^2) + \lambda_2(lx+my+nz) = 0$$

$$2U + 2\lambda_1(1) + \lambda_2(0) = 0$$

$$2U + 2\lambda_1 = 0$$

$$\boxed{\lambda_1 = -U}$$

$$2x - 2aUx + \lambda_2 l = 0$$

$$2x(1-aU) + \lambda_2 l = 0$$

$$2x(1-aU) = -\lambda_2 l$$

$$\boxed{x = \frac{-\lambda_2 l}{2(1-aU)}}$$

lly $\boxed{y = \frac{-\lambda_2 m}{2(1-bU)}}$

lly $\boxed{z = \frac{-\lambda_2 n}{2(1-cU)}}$

x, y, z value sub in eq (3)

$$lx + my + nz = 0$$

$$l\left(\frac{-\lambda_2 l}{2(1-aU)}\right) + m\left(\frac{-\lambda_2 m}{2(1-bU)}\right) + n\left(\frac{-\lambda_2 n}{2(1-cU)}\right) = 0$$

$$-\frac{\lambda^2}{2}\left[\frac{l^2}{1-aU} + \frac{m^2}{1-bU} + \frac{n^2}{1-cU}\right] = 0$$

(8) Find the max value of $f(x,y,z) = x^a y^b z^c$ subject to the condition $x+y+z=1$

Given, $f(x,y,z) = x^a y^b z^c$ — (1)

$x+y+z=1$ — (2)

let $\phi(x,y,z) = x+y+z-1$ — (3)

lagrange's function

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$= x^a y^b z^c + \lambda(x+y+z-1)$$

$$\frac{\partial F}{\partial x} = y^b z^c (a x^{a-1}) + \lambda \quad \text{— (4)}$$

$$\frac{\partial F}{\partial y} = x^a z^c (b y^{b-1}) + \lambda \quad \text{— (5)}$$

$$\frac{\partial F}{\partial z} = x^a y^b (c z^{c-1}) + \lambda \quad \text{— (6)}$$

for mini or maxi

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial z} = 0$$

$a x^{a-1} y^b z^c + \lambda = 0$ — (7)

$b x^a y^{b-1} z^c + \lambda = 0$ — (8)

$c x^a y^b z^{c-1} + \lambda = 0$ — (9)

x eq^n (7) + $y \cdot$ eq^n (8) + z eq^n (9)

$$ax^a y^b z^c + bx^a y^b z^c + cx^a y^b z^c + \lambda(x+y+z) = 0$$

$$x^a y^b z^c (a+b+c) + \lambda(x+y+z) = 0$$

$$\lambda = -(a+b+c)x^a y^b z^c$$

λ value sub in eq(3)

$$a \cdot \frac{x^a}{x} y^b z^c + (-(a+b+c))(x^a y^b z^c) = 0$$

$$a \cdot \frac{x^a y^b z^c}{x} = x^a y^b z^c (a+b+c)$$

$$\frac{a}{x} = a+b+c$$

$$\frac{x}{a} = \frac{1}{a+b+c}$$

llly $\quad y = \dfrac{b}{a+b+c}$

$$z = \frac{c}{a+b+c}$$

sub x, y, z value in eq(1)

$$x+y+z = 1$$

$$\frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1$$

$$\frac{a+b+c}{a+b+c} = 1$$

sub x, y, z in eq (1)

max value $x^a y^b z^c$

$$= \left(\frac{a}{a+b+c}\right)^a \cdot \left(\frac{b}{a+b+c}\right)^b \left(\frac{c}{a+b+c}\right)^c$$

$$= \frac{a^a b^b c^c}{(a+b+c)^{a+b+c}}$$

(7) find the volume of the greatest parallel piped incribed in the ellipsoid where eq : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

suppose that dimensions of parallelpiped are

$$(2x)(2y)(2z) = lbh$$

$$v = (2x)(2y)(2z)$$

$$v = 8xyz \quad -(1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad -(2)$$

$$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

lagranges function

$$f(x, y, z) = f(v) + \lambda \phi$$

$$= 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\frac{\partial f}{\partial x} = 8yz + \frac{2x\lambda}{a^2} \quad -(3)$$

$$\frac{\partial f}{\partial y} = 8xz + \frac{2y\lambda}{b^2} \quad -(4)$$

$$\frac{\partial f}{\partial z} = 8xy + \frac{2z\lambda}{c^2} \quad -(5)$$

for maxi or mini

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial z} = 0$$

$$8yz + \frac{2x}{a^2}\lambda = 0 \quad -(6)$$

$$8xz + \frac{2y}{b^2}\lambda = 0 \quad -(7)$$

$$8xy + \frac{2z}{c^2}\lambda = 0 \quad -(8)$$

eq(1) x + eq(2) $\cdot y$ + eq(3) $\cdot z$

$$8yz + \frac{2x^2\lambda}{a^2} + 8xyz + \frac{2y^2\lambda}{b^2} + 8yzx + \frac{2z^2\lambda}{c^2} = 0$$

$$8xyz(1+1+1) + 2\lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0$$

$$24xyz + 2\lambda = 0$$

$$-24xyz = 2\lambda$$

$$\boxed{\lambda = -12xyz}$$

sub λ in eq(3)

$$8xyz + \frac{2x(-12xyz)}{a^2} = 0$$

$$8a^2yz - 24x^2yz = 0$$

$$\cancel{8}a^2\cancel{yz} = \overset{3}{\cancel{24}}x^2\cancel{yz}$$

$$\frac{a^2}{3} = x^2$$

$$x = \frac{a}{\sqrt{3}}$$

lly $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$

sub x, y, z in eq (1)

$$v = 8xyz$$

$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$\boxed{V = \frac{8abc}{3\sqrt{3}}}$$ //

Equality of two functions:

$$f_{xy}(a, b) = \lim_{h \to 0} \frac{f_y(a+h, b) - f_y(a, b)}{h}$$

$$f_y(a+h, b) = \lim_{k \to 0} \frac{f(a+h, b+k) - f(a+h, b)}{k}$$

$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

$$f_{yx}(a, b) = \lim_{k \to 0} \frac{f_x(a, b+k) - f_x(a, b)}{k}$$

$$f_x(a, b+k) = \lim_{h \to 0} \frac{f(a+h, b+k) - f(a, b+k)}{h}$$

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Example: Prove that $f_{xy}(0,0) \neq f_{yx}(0,0)$ for the function f is given by

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0)$$

$$f(0, 0) = 0$$

Sol

$$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} \quad \text{--- (1)}$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{hk(h^2-k^2)}{h^2+k^2} - 0}{k}$$

$$= \lim_{k\to 0} \frac{h\cancel{k}(h^2-k^2)}{\cancel{k}(h^2+k^2)}$$

$$= \frac{h(h^2-0)}{h^2+0}$$

$$f_y(h,0) = h \quad \text{--- (2)}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k}.$$

$$f_y(0,0) = 0 \quad \text{--- (3)}$$

from eq (1)

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h}$$

$$f_{xy}(0,0) = 1 \quad \text{--- (4)}$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \quad \text{--- (5)}$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{hk(h^2-k^2)}{h^2+k^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}k(h^2-k^2)}{\cancel{h}(h^2+k^2)}$$

$$= \frac{k(0-k^2)}{0+k^2}$$

$$= \frac{-k^3}{k^2}$$

$$f_x(0,k) = -k \quad \text{—} \quad (5)$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h}$$

$$f_x(0,0) = 0 \quad \text{—} \quad (4)$$

from eq (5)

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-k-0}{k}$$

$$= \lim_{k\to 0} -1$$

$$= -1$$

$f_{yx}(0,0) = -1$ — (8)

equating eq (4) & (8)

$f_{xy}(0,0) \neq f_{yx}(0,0)$

Hence proved.

(7) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$ where $f(x,y)=0$ if $xy=0$, $f(x,y) = x^2 \, Tan^{-1}\left(\frac{y}{x}\right) - y^2 \cdot Tan^{-1}\left(\frac{x}{y}\right)$ if $xy \neq 0$

Given that, $f(x,y)=0$ if $xy=0$

$f(x,y) = x^2 \, Tan^{-1}\left(\frac{y}{x}\right) - y^2 \, Tan^{-1}\left(\frac{x}{y}\right)$ if $xy \neq 0$

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \quad \text{— (1)}$$

$$f_y(h,0) = \lim_{k \to 0} \frac{f(h, 0+k) - f(h,0)}{k}$$

$$= \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k \to 0} \frac{h^2 \, Tan^{-1}\left[\frac{k}{h}\right] - k^2 \, Tan^{-1}\left[\frac{h}{k}\right] - 0}{k}$$

$$= \lim_{k \to 0} \frac{h^2 \, Tan^{-1}\left(\frac{k}{h}\right)}{k} - \frac{k^{\cancel{2}} \, Tan^{-1}\left[\frac{h}{k}\right]}{\cancel{k}}$$

$$= \lim_{k\to 0} \frac{h \operatorname{Tan}^{-1}\left(\frac{k}{h}\right)}{\left(\frac{k}{h}\right)} - \lim_{k\to 0} k \cdot \operatorname{Tan}^{-1}\left(\frac{h}{k}\right)$$

$$= h(1) - 0.$$

$$f_y(h, 0) = h \text{ --- } (2)$$

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, 0+k) - f(0, 0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k}$$

$$f_y(0, 0) = 0 \text{ --- } (3)$$

sub (2) & (2) in eq (1)

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{h - 0}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}}$$

$$f_{xy}(0, 0) = 1 \text{ --- } (4)$$

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f_x(0, 0+k) - f_x(0, 0)}{k}$$

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} \text{ --- } (5)$$

$$f_x(0, k) = \lim_{h\to 0} \frac{f(0+h, k) - f(0, k)}{h}$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \tan^{-1}\left[\frac{k}{h}\right] - k^2 \tan^{-1}\left[\frac{h}{k}\right] - 0}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \tan^{-1}\left(\frac{k}{h}\right)}{\cancel{h}} - \lim_{h\to 0} \frac{k^2 \tan^{-1}\left[\frac{h}{k}\right]}{h}$$

$$= \lim_{h\to 0} h \tan^{-1}\left(\frac{k}{h}\right) - \lim_{h\to 0} \frac{k \tan^{-1}\left[\frac{h}{k}\right]}{\left(\frac{h}{k}\right)}$$

$$= 0 - k(1)$$

$$= -k$$

$$f_x(0,k) = -k \quad \text{—} \ (6)$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h}$$

$$f_x(0,0) = 0 \quad \text{—} \ (7)$$

sub (6) & (7) in (5)

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k}$$

$$= \lim_{k \to 0} \frac{-\cancel{k}}{\cancel{k}}$$

$$f_{yx}(0,0) = -1 \quad \text{——} \; (8)$$

equating eq (4) & (8)

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$