UNIT-3 CURVATURE AND EVOLUTES

Radius of curvature :-

The reciprocal of the curvature of a curve at any point is called its radius of the curvature at the point. It is denoted by ρ

$$\rho = \frac{ds}{d\psi}$$

Find the radius of the curvature at any point of the following.

1. $s = c \tan \psi$

D wrt ψ

$$\frac{ds}{d\psi} = c \sec^2 \psi$$

$$\rho = c \sec^2 \psi$$

2. $s = 4a \sin \psi$

D wrto ψ

$$\frac{ds}{d\psi} = 4a \cos \psi$$

$$\rho = 4a \cos \psi$$

3. $s = 4a \sin \frac{1}{3} \psi$

$$\frac{ds}{d\psi} = 4a \cos \frac{1}{3} \psi \cdot \frac{1}{3}$$

$$= \frac{4a}{3} \cos \frac{1}{3} \psi$$

4. $s = c\log(\sec\psi)$

$$\frac{ds}{d\psi} = c\,\frac{1}{\sec\psi}\cdot\sec\psi\tan\psi$$

$$\rho = c\tan\psi$$

5. $s = a\log(\tan\psi + \sec\psi) + a\tan\psi\sec\psi$

$$\frac{ds}{d\psi} = \frac{a}{\tan\psi + \sec\psi}\sec^2\psi + \sec\psi\tan\psi + a[\sec^2\psi\sec\psi + \tan\psi\sec\psi\tan\psi]$$

$$= \frac{a\sec\psi[\sec\psi + \tan\psi]}{\tan\psi + \sec\psi} + a\sec^3\psi + a\sec\psi\tan^2\psi$$

$$= a\sec\psi + a\sec^3\psi + a\sec\psi\tan^2\psi$$

$$= a\sec^3\psi + a\sec^3\psi$$

$$= 2a\sec^3\psi$$

Length of arc as a function, Derivative of arc

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

1. find the $\frac{ds}{dx}$ for the curve.

(a) $y = \cosh\left(\frac{x}{c}\right)$

D w.r.t. x

$$\frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)\cdot\frac{d}{dx}\left(\frac{x}{c}\right)$$

$= \sinh\left(\frac{x}{c}\right) \cdot \frac{1}{c}$

$\frac{dy}{dx} = \frac{1}{c} \cdot \sinh\left(\frac{x}{c}\right)$

$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$= \sqrt{1 + \left[\frac{1}{c} \sinh\left(\frac{x}{c}\right)\right]^2}$

$= \sqrt{1 + \frac{1}{c^2} \sinh^2\left(\frac{x}{c}\right)}$

(b) $y = a \log\left[\frac{a^2}{a^2 - x^2}\right]$

D w.r.to x

$\frac{dy}{dx} = a \cdot \frac{1}{\frac{a^2}{a^2 - x^2}} \cdot \frac{d}{dx}\left(\frac{a^2}{a^2 - x^2}\right)$

$= \frac{a^2 - x^2}{\cancel{a}} \cdot \cancel{a^2}\left[\frac{-1}{(a^2 - x^2)^2}(-2x)\right]$

$= \cancel{a^2 - x^2} \; a \frac{2x}{(a^2 - x^2)^{\cancel{2}}}$

$\frac{dy}{dx} = \frac{2ax}{a^2 - x^2}$ //

(2) S.T for the parametric equation $x = f(t)$ $y = f(t)$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ and use this result find $\frac{ds}{dt}$ for

(i) $x = a\cos^3 t$, $y = b\sin^3 t$

(ii) $x = a(t - \sin t)$, $y = a(1 - \cos t)$

(iii) $x = ae^t \sin t$, $y = ae^t \cos t$

(iv) $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

Given parametric equation are $x = f(t)$, $y = F(t)$

$$\frac{dx}{dt} = f'(t) \qquad \frac{dy}{dt} = F'(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{F'(t)}{f'(t)}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1 + \frac{(F'(t))^2}{(f'(t))^2}}$$

$$= \frac{\sqrt{f'(t)^2 + F(t)^2}}{f'(t)}$$

$$\frac{ds}{dx} f'(t) = \sqrt{f'(t)^2 + (F'(t))^2}$$

$$\frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\boxed{\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

(i) $x = a\cos^3 t = a(\cos t)^3$ $y = b(\sin t)^3$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$\frac{dx}{dt} = a\, 3\cos^2 t(-\sin t)$ $\frac{dy}{dt} = b\, 3\sin^2 t(\cos t)$

$\frac{dx}{dt} = -3a\cos^2 t\sin t$ $\frac{dy}{dt} = 3b\sin^2 t\cos t$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-3a\cos^2 t\sin t)^2 + (3b\sin^2 t\cos t)^2}$$

$$= \sqrt{9a^2\cos^4 t\sin^2 t + 9b^2\sin^4 t\cos^2 t}$$

$$= \sqrt{9\cos^2 t\sin^2 t\left[a^2\cos^2 t + b^2\sin^2 t\right]}$$

$$\frac{ds}{dt} = 3\cos t\sin t\sqrt{a^2\cos^2 t + b^2\sin^2 t} \;//$$

(ii) $x = a(t - \sin t)$ $y = a(1 - \cos t)$

$\frac{dx}{dt} = a(1 - \cos t)$ $\frac{dy}{dx} = a(\sin t)$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t}$$

$$= \sqrt{a^2(2\sin^2 t/_2)^2 + a^2 \sin^2 t}$$

$$= \sqrt{a^2(4\sin^4 t/_2) + a^2(2\sin t/_2 \cos t/_2)^2}$$

$$= \sqrt{4a^2 \sin^2 t/_2 \left[\sin^2 t/_2 + \cos^2 t/_2\right]}$$

$$\frac{ds}{dt} = 2a \sin t/_2 //$$

(iii) $x = ae^t \sin t \qquad y = ae^t \cos t$

$$\frac{dx}{dt} = a(e^t \sin t + e^t \cos t) \qquad \frac{dy}{dt} = a\left[e^t \cos t - \sin t\, e^t\right]$$

$$\frac{dx}{dt} = ae^t(\sin t + \cos t) \qquad \frac{dy}{dt} = ae^t\left[\cos t - \sin t\right]$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{a^2 e^{2t}(\sin t + \cos t)^2 + a^2 e^{2t}(\cos t - \sin t)^2}$$

$$= \sqrt{a^2 e^{2t}\left[\sin^2 t + \cos^2 t + 2\sin t \cos t + \cos^2 t + \sin^2 t - 2\cos t \sin t\right.}$$

$= \sqrt{a^2 e^{2t}(1+1)}$

$\frac{ds}{dt} = \sqrt{2}\, a e^t$ //

(iv) $x = a(\cos t + \sin t)$ $\quad y = a(\sin t - \cancel{t}\cos t)$

$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$ $\quad \frac{dy}{dt} = a(\cos t - \cos t + t\sin t)$

$\frac{dy}{dt} = at\sin t$

$\frac{dx}{dt} = at\cos t$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{a^2 t^2 \cos^2 t + a^2 t^2 \sin^2 t}$$

$$= \sqrt{a^2 t^2 (\sin^2 t + \cos^2 t)}$$

$$= \sqrt{a^2 t^2 (1)}$$

$\frac{ds}{dt} = at$ //

Note:

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

$1 + \cos 2\theta = 2\cos^2\theta$

$1+\cos\theta = 2\cos^2 \theta/2$

$1-\cos 2\theta = 2\sin^2\theta$

$1-\cos\theta = 2\sin^2 \theta/2$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin \theta/2 \cos\theta/2$

$1+\tan^2\theta = \sec^2\theta$

(1) S.T for polar equation $r=f(\theta)$ $\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$

The polar equation are $x = r\cos\theta$, $y = r\sin\theta$

$x = r\cos\theta$

$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta \cdot \frac{dr}{d\theta}$

$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\, r_1$

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + \sin\theta \cdot \frac{dr}{d\theta}$

$\frac{dy}{d\theta} = r\cos\theta + \sin\theta\, r_1$

Here, $\boxed{r_1 = \frac{dr}{d\theta}}$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\, r_1}{r_1\cos\theta - r\sin\theta}$

$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\sqrt{1+\frac{(r\cos\theta+r_1\sin\theta)^2}{(r_1\cos\theta-r\sin\theta)^2}}$$

$$\sqrt{\frac{(r_1\cos\theta-r\sin\theta)^2+(r\cos\theta+r_1\sin\theta)^2}{(r_1\cos\theta-r\sin\theta)^2}}$$

$$\frac{\sqrt{r_1^2\cos^2\theta+r^2\sin^2\theta-2r_1r\cos\theta\sin\theta+r^2\cos^2\theta+r_1^2\sin^2\theta+2rr_1\cos\theta\sin\theta}}{(r_1\cos\theta-r\sin\theta)}$$

$$\frac{ds}{dx}=\frac{\sqrt{r_1^2(\cos^2\theta+\sin^2\theta)+r^2(\cos^2\theta+\sin^2\theta)}}{\frac{dx}{d\theta}}$$

$$\frac{ds}{x}\cdot\frac{dx}{d\theta}=\sqrt{r_1^2+r^2}$$

$$\frac{ds}{dx}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\ //$$

find $\frac{ds}{d\theta}$ to the following given $r=a(1+\cos\theta)$

$$r=a(1+\cos\theta)$$

D wrto θ

$$\frac{dr}{d\theta}=a(-\sin\theta)$$

$$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$=\sqrt{r^2+a^2\sin^2\theta}$$

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{a^2(1+\cos^2\theta + 2\cos\theta + \sin^2\theta)}$$

$$= a\sqrt{2+2\cos\theta}$$

$$= a\sqrt{2(1+\cos\theta)}$$

$$= a\sqrt{2(2\cos^2\theta/2)}$$

$$\frac{ds}{d\theta} = 2a\cos\theta/2$$

(ii) $r^2 = a^2\cos 2\theta$

$$r = \sqrt{a^2\cos 2\theta}$$

D wrt θ

$$\frac{dr}{d\theta} = \frac{1}{\cancel{2}\sqrt{a^2\cos 2\theta}}\left(-a^2\sin 2\theta(\cancel{2})\right)$$

$$= \frac{-a^2\sin 2\theta}{\sqrt{a^2\cos 2\theta}}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2\cos 2\theta + \frac{a^4\sin^2 2\theta}{a^2\cos 2\theta}}$$

$$= \sqrt{\frac{a^4\cos^2 2\theta + a^4\sin^2 2\theta}{a^2\cos 2\theta}}$$

$$= \frac{a^2\sqrt{\cos^2\theta + \sin\theta}}{a\sqrt{\cos 2\theta}}$$

$$= \frac{a}{\sqrt{\cos 2\theta}} \,//$$

Eg: In the curve $r^m = a^m \cos(m\theta)$ Prove that

(i) $\frac{ds}{d\theta} = a \sec^{\frac{m-1}{m}}(m\theta)$

(ii) $a^{2m}\frac{d^2r}{ds^2} + m \cdot r^{2m-1} = 0$

Sol: Given $r^m = a^m \cos(m\theta)$ — (1)

$$r = a\left(\cos(m\theta)\right)^{1/m}$$

Apply log on b.s

$$\log(r^m) = \log\left(a^m \cos(m\theta)\right)$$

D wrt θ

$$\cancel{m} \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta}\left(-\sin(m\theta)\right)\cancel{m}$$

$$\boxed{\frac{dr}{d\theta} = -r\tan(m\theta)}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{r^2 + r^2 \tan^2(m\theta)}$$

$$= \sqrt{r^2(1 + \tan^2 m\theta)}$$

$$= \sqrt{r^2 \sec^2 m\theta}$$

$$\frac{ds}{d\theta} = r\sec m\theta$$

$$= a\left(\cos(m\theta)\right)^{1/m} \sec(m\theta)$$

$$= a\sec m\theta \cdot \frac{1}{(\sec m\theta)^{1/m}}$$

$$= a\left(\sec(m\theta)\right)^{1-\frac{1}{m}}$$

$$\boxed{\frac{ds}{d\theta} = a \cdot \sec^{\frac{m-1}{m}}(m\theta)}$$

(ii) $\frac{ds}{d\theta} = r\sec(m\theta)$

$\frac{ds}{d\theta} = \frac{r}{\cos(m\theta)}$ from eq(1)

$$\frac{ds}{d\theta} = \frac{r}{\frac{r^m}{a^m}}$$

$$= a^m \cdot \frac{r}{r^m}$$

$$\boxed{\frac{ds}{d\theta} = a^m \cdot r^{1-m}}$$

$$\frac{dr}{ds} = \frac{dr}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= -r\,\mathrm{Tan}(m\theta) \cdot \frac{1}{a^m \cdot r^{1-m}}$$

$$\frac{dr}{ds} = -a^{-m} r^m\, \mathrm{Tan}(m\theta)$$

$$\frac{d^2r}{ds^2} = \frac{d}{ds}\left(\frac{dr}{ds}\right)$$

$$= \left(\frac{d}{dr} \cdot \frac{dr}{ds}\right)\left(\frac{dr}{ds}\right)$$

$$= \frac{dr}{ds} \cdot \frac{d}{dr}\left(\frac{dr}{ds}\right)$$

$$= -a^{-m} r^m \cdot \mathrm{Tan}(m\theta) \cdot \frac{d}{dr}\left[-a^{-m} r^m\, \mathrm{Tan}(m\theta)\right]$$

$$= \left[-a^{-m} r^m\, \mathrm{Tan}(m\theta)\right]\left[(-a^{-m})\left(\mathrm{Tan}(m\theta)\, m r^{m-1} + r^m \cdot \sec^2 m\theta \cdot m \cdot \frac{d\theta}{dr}\right)\right]$$

$$= -a^{-2m} \cdot r^m\, \mathrm{Tan}(m\theta) \cdot r^m \cdot m \left[\frac{\mathrm{Tan}(m\theta)}{r} + \frac{\sec^2(m\theta)}{-r\,\mathrm{Tan}(m\theta)}\right]$$

$$\frac{d^2r}{ds^2} = \frac{a^{-2m} r^{2m} \cdot m \cdot \mathrm{Tan}(m\theta)}{-r}\left[\frac{\sec^2(m\theta)}{\mathrm{Tan}(m\theta)} - \mathrm{Tan}(m\theta)\right]$$

$$= -a^{-2m} r^{2m-1} \cdot m\, \mathrm{Tan}(m\theta)\left[\frac{\sec^2 m\theta - \mathrm{Tan}^2(m\theta)}{\mathrm{Tan}(m\theta)}\right]$$

$$\frac{d^2r}{ds^2} = -a^{-2m} \cdot r^{2m-1} \cdot (m)(1)$$

$$a^{2m} \cdot \frac{d^2r}{ds^2} = -m r^{2m-1}$$

$$a^{2m} \cdot \frac{d^2 r}{ds^2} + m r^{2m-1} = 0$$

Radius of curvature:

Radius of the curvature - cartesian equation:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Radius of the curvature - polar equation

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$$\text{Curvature} = \frac{1}{\rho}$$

Radius of the curvature - parametric equation

$x = f(t) \qquad y = F(t)$

$$\rho = \pm \frac{\left[f'(t)^2 + (F'(t))^2\right]^{3/2}}{f'(t) F''(t) - F'(t) f''(t)}$$

**imp

Example: for the cycloid

$$x = a(t + \sin t), \quad y = a(1 - \cos t)$$

Given,

$$x = a(t + \sin t) \qquad y = a(1 - \cos t)$$

D w.r.to t

$$\frac{dx}{dt} = a[1 + \cos t] \qquad \frac{dy}{dt} = a[0 + \sin t]$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{a} \sin t}{\cancel{a}[1+\cos t]}$$

$$= \frac{\cancel{2} \sin(t/2) \cos(t/2)}{\cancel{2} \cos^2(t/2)}$$

$$= \frac{\sin(t/2)}{\cos(t/2)}$$

$$\frac{dy}{dx} = \tan(t/2)$$

D w.r.to x

$$\frac{d^2y}{dx^2} = \sec^2(t/2) \cdot \frac{d}{dx}\left(\frac{t}{2}\right)$$

$$= \sec^2(t/2) \cdot \frac{1}{2} \cdot \frac{dt}{dx}$$

$$= \sec^2(t/2) \cdot \frac{1}{2} \cdot \frac{1}{a(1+\cos t)}$$

$$= \sec^2(t/2) \cdot \frac{1}{2a} \cdot \frac{1}{2\cos^2(t/2)}$$

$$= \frac{1}{4a} \sec^2(t/2) \cdot \frac{1}{\cos^2(t/2)}$$

$$= \frac{1}{4a} \sec^2(t/2) \cdot \sec^2(t/2)$$

$$\frac{d^2y}{dx^2} = \frac{\sec^4(t/2)}{4a}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1 + [\tan(t/2)]^2\right]^{3/2}}{\frac{\sec^4(t/2)}{4a}}$$

$$= \frac{4a\left(1 + \tan^2(t/2)\right)^{3/2}}{\sec^4(t/2)}$$

$$= \frac{4a\left(\sec^{\cancel{2}}(t/2)\right]^{3/\cancel{2}}}{\sec^4(t/2)}$$

$$= \frac{4a \sec^3(t/2)}{\sec^{\cancel{4}}(t/2)}$$

$$\Rightarrow 4a \frac{1}{\sec(t/2)} = 4a \cos(t/2)$$

Hence proved //

Example 2: for the curve

$r^m = a^m \cos(m\theta)$ P.T, $\rho = \dfrac{a^m}{(m+1) r^{m-1}}$

Given, $r^m = a^m \cos(m\theta)$ — (1)

Apply log on bs

$\log r^m = \log(a^m \cos(m\theta))$

$m \log r = \log a^m + \log \cos(m\theta)$

$m \log r = m \log a + \log(\cos m\theta)$

D w.r.t θ

$m \cdot \dfrac{1}{r} \cdot \dfrac{dr}{d\theta} = 0 + \dfrac{1}{\cos m\theta}(-\sin m\theta) m$

$\dfrac{\cancel{m}}{r} \cdot \dfrac{dr}{d\theta} = -\cancel{m} \cdot \tan(m\theta)$

$r_1 = \dfrac{dr}{d\theta} = -r \tan(m\theta)$ — (2)

D w.r.t θ

$\dfrac{d^2 r}{d\theta^2} = -\left\{ r \sec^2(m\theta) \cdot m + \tan(m\theta) \cdot \dfrac{dr}{d\theta} \right\}$

$r_2 = \dfrac{d^2 r}{d\theta^2} = -\left\{ m r \sec^2(m\theta) + \tan(m\theta)(-r \tan(m\theta)) \right\}$

$$\frac{d^2r}{d\theta^2} = -mr\sec^2(m\theta) + r\cdot\tan^2(m\theta) \quad \text{— (3)}$$

Radius of curvature $\left.\right\}$ $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$$= \frac{(r^2+(-r\tan m\theta)^2)^{3/2}}{r^2+2(-r\tan m\theta)^2 - r(-mr\sec^2(m\theta)+r\cdot\tan^2(m\theta))}$$

$$= \frac{(r^2+r^2\tan^2 m\theta)^{3/2}}{r^2+2r^2\tan^2(m\theta)+mr^2\sec^2(m\theta)-r^2\tan^2(m\theta)}$$

$$= \frac{(r^2(1+\tan^2 m\theta))^{3/2}}{r^2+r^2\tan^2(m\theta)+mr^2\sec^2(m\theta)}$$

$$= \frac{(r^2\sec^2(m\theta))^{3/2}}{r^2(1+\tan^2(m\theta)+mr^2\sec^2(m\theta)}$$

$$= \frac{r^3\cdot\sec^3(m\theta)}{r^2(\sec^2(m\theta))+mr^2\sec^2(m\theta)}$$

$$= \frac{r^3\sec^3(m\theta)}{r^2\sec^2(m\theta)(1+m)} = \frac{r\sec(m\theta)}{1+m}$$

$$= \frac{r \cdot \sec(m\theta)}{1+m}$$

$$= \frac{r \cdot \frac{1}{\cos(m\theta)}}{1+m}$$

$$= \frac{r \cdot \frac{\frac{1}{r^m}}{a^m}}{1+m}$$

$$= \frac{r \cdot \frac{a^m}{r^m}}{1+m}$$

$$= \frac{r \cdot a^m \cdot r^{-m}}{1+m}$$

$$= \frac{r^{1-m} \cdot a^m}{1+m} \qquad \frac{a^m}{r^{m-1}}$$

$$\rho = \frac{a^m}{(1+m)\, r^{m-1}} \; //$$

Hence proved.

Imp

Examples

1) S.T the curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on folium function $x^3 + y^3 = 3axy$ is $\frac{-8\sqrt{2}}{3a}$

$$x^3 + y^3 = 3axy$$

D w.r.to x

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3a\left(y + x \cdot \frac{dy}{dx}\right)$$

$$3x^2+3y^2\frac{dy}{dx}=3ay+3ax\cdot\frac{dy}{dx}$$

$$\cancel{3}\frac{dy}{dx}(y^2-ax)=\cancel{3}(ay-x^2) \quad -(1)$$

$$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$$

At the point $\left(\frac{3a}{2},\frac{3a}{2}\right)$

$$\frac{dy}{dx}=\frac{a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)}$$

$$=\frac{\frac{3a^2}{2}-\frac{9a^2}{4}}{\frac{9a^2}{4}-\frac{3a^2}{2}}$$

$$=\frac{\frac{6a^2-9a^2}{\cancel{4}}}{\frac{9a^2-6a^2}{\cancel{4}}}$$

$$=\frac{-3a^2}{3a^2}$$

$$\frac{dy}{dx}=-1$$

from eq(1)

$$\frac{dy}{dx}(y^2-ax)=(ay-x^2)$$

Dwrto x

$$\frac{dy}{dx}\left(2y\cdot\frac{dy}{dx}-a\right)+(y^2-ax)\frac{d^2y}{dx^2}=\left(a\frac{dy}{dx}-2x\right)$$

At $P\left(\frac{2a}{2}, \frac{3a}{2}\right)$

$$(-1)\left[3a(-1)-a\right]+\left[\frac{9a^2}{4}-\frac{3a^2}{2}\right]\cdot\frac{d^2y}{dx^2}=\left[a(-1)-3a\right]$$

$$4a+\left(\frac{3a^2}{4}\right)\cdot\frac{d^2y}{dx^2}=-4a$$

$$\frac{3a^2}{4}\cdot\frac{d^2y}{dx^2}=-4a-4a$$

$$\frac{3a^2}{4}\cdot\frac{d^2y}{dx^2}=-8a$$

$$\frac{d^2y}{dx^2}=-8a\cdot\frac{4}{3a^2}$$

$$\frac{d^2y}{dx^2}=\frac{-32}{3a}$$

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$=\frac{\left(1+(-1)^2\right)^{3/2}}{\frac{-32}{3a}}$$

$$=\frac{-3a(2)^{3/2}}{32}$$

$$\rho=\frac{-3a(8)^{1/2}}{32}$$

$$\rho=\frac{\sqrt{8}(-3a)}{32}$$

<u>Curvature</u>: $\frac{1}{\rho}=\frac{-32}{3a\sqrt{8}}$

$$=\frac{-8\times4}{3a\sqrt{8}}$$

$$=\frac{-\sqrt{8}\cdot\cancel{\sqrt{8}}\times4}{3a\cancel{\sqrt{8}}}$$

$$=\frac{-\sqrt{2\times4}\cdot4}{3a}$$

$$=\frac{-2\sqrt{2}\cdot4}{3a}$$

Curvature $:-\frac{8\sqrt{2}}{3a}$ //

find ρ for the following

(1) $y = c \cosh\left(\frac{x}{c}\right)$

D w.r.to x

$$\frac{dy}{dx} = c \cdot \sinh\left(\frac{x}{c}\right) \cdot \frac{d}{dx}\left(\frac{x}{c}\right)$$

$$= \cancel{c} \cdot \sinh\left(\frac{x}{c}\right)\left(\frac{1}{\cancel{c}}\right)$$

$$\frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)$$

D w.r.to x

$$\frac{d^2y}{dx^2} = \cosh\left(\frac{x}{c}\right) \cdot \frac{1}{c}$$

$$= \frac{1}{c} \cos(h)\left(\frac{x}{c}\right)$$

$$\boxed{1 + \sinh^2 x = \cosh^2 x}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1 + \sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)}$$

$$= \frac{c\left[\cosh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh(x/c)}$$

$$= \frac{c \cdot \cosh^3\left(\frac{x}{c}\right)}{\cosh\left(\frac{x}{c}\right)}$$

$$= c\cdot \cosh^2\left(\frac{x}{c}\right)$$

M & D by c

$$= c\cdot \cosh^2 \cdot \frac{x}{c}\cdot \frac{c}{c}$$

$$= \frac{c^2\cosh^2\left(\frac{x}{c}\right)}{c}$$

$$= \frac{\left[c\cosh\left(\frac{x}{c}\right)\right]^2}{c}$$

$$\rho = \frac{y^2}{c}$$ //

(2) $x = a[\cos t + t\sin t]$, $y = a[\sin t - t\cos t]$

$$\frac{dx}{dt} = a[-\sin t + t\cos t + \sin t]$$

$$\frac{dy}{dx} = a[\cos t - \cos t + t\sin t]$$

$$\frac{dy}{dx} = at\sin t$$

$$\frac{dx}{dt} = at\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{at\sin t}{at\cos t}$$

$$\frac{dy}{dx} = \tan t$$

D w.r.t. x

$$\frac{d^2y}{dx^2} = \sec^2 t\cdot \frac{dt}{dx}$$

$$= \sec^2 t \cdot \frac{1}{a t \cos t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{at} \cdot \sec^3 t$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1+\tan^2 t\right)^{3/2}}{\frac{1}{at}\sec^3 t}$$

$$= \frac{\left(\sec^{\cancel{2}} t\right)^{3/\cancel{2}}}{\frac{1}{at}\sec^3 t}$$

$$= \frac{\cancel{\sec^3 t}}{\frac{1}{at}\cancel{\sec^3 t}}$$

$$\rho = at \;//$$

Example:

(*) Find the radius of the curvature for the curve

$$r = a(1-\cos\theta)$$

$$r = a(1-\cos\theta)$$

D w.r.to θ

$$r_1 = \frac{dr}{d\theta} = a\sin\theta$$

D w.r.to θ

$$r_2 = \frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{\left(a^2(1-\cos\theta)^2+a^2\sin^2\theta\right)^{3/2}}{a^2(1-\cos\theta)^2+2a^2\sin^2\theta-a^2(\cos\theta-\cos^2\theta)}$$

$$= \frac{\left(a^2(1+\cos^2\theta-2\cos\theta)+a^2\sin^2\theta\right)^{3/2}}{a^2(1+\cos^2\theta-2\cos\theta)+2\sin^2\theta.a^2-a^2\cos\theta+a^2\cos^2\theta}$$

$$= \frac{\left(a^2+a^2\cos^2\theta-2a^2\cos\theta+a^2\sin\theta\right)^{3/2}}{a^2[1+\cos^2\theta-2\cos\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]}$$

$$= \frac{\left(a^2-a^2\cos\theta+a^2\sin\theta\right)^{3/2}}{a^2[1+2\cos^2\theta+2\sin^2\theta-3\cos\theta]}$$

$$= \frac{a^2[1-\cos\theta+\sin\theta]^{3/2}}{a^2[1+2(1)-3\cos\theta]}$$

$$= \frac{\left(a^2+a^2\cos^2\theta-2a^2\cos\theta+a^2\sin\theta\right)^{3/2}}{a^2(1+2(1)-3\cos\theta)}$$

$$= \frac{a^3(1+1-2\cos\theta)^{3/2}}{a^2(3-3\cos\theta)}$$

$$= \frac{a\left(2(1-\cos\theta)\right)^{3/2}}{3(1-\cos\theta)} \qquad = \frac{\sqrt{a}\,\sqrt{a}\,2\sqrt{2}}{3\sqrt{a}}\cdot\sqrt{r}$$

$$= \frac{a\sqrt{8}\,(1-\cos\theta)^{3/2-1}}{3} \qquad \boxed{\rho = \frac{2}{3}\sqrt{2ar}}$$

$$= \frac{a\sqrt{8}}{3}\sqrt{\frac{r}{a}}$$

(*) P.T for the curve $r = a(1+\cos\theta)$ $\frac{\rho^2}{r}$ is constant.

Given $r = a(1+\cos\theta)$

D w.r.to θ

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta$$

D w.r.t θ

$$r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2(\sin^2\theta)\right]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2(\cos\theta + \cos^2\theta)}$$

$$= \frac{\left(a^2(1+\cos^2\theta + 2\cos\theta) + a^2\sin^2\theta\right)^{3/2}}{a^2\left[1+\cos^2\theta + 2\cos\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta\right]}$$

$$= \frac{a^3(1+1+2\cos\theta)^{3/2}}{a^2[1+2(1)+3\cos\theta]}$$

$$= \frac{a^3[2^{3/2}][1+\cos\theta]^{3/2}}{a^2(1+2+3\cos\theta]}$$

$$= \frac{a\sqrt{8}}{3[1+\cos\theta]}(1+\cos\theta)^{3/2}$$

$$= \frac{a\sqrt{8}}{3}(1+\cos\theta)^{1/3}$$

$$= \frac{a\sqrt{8}}{3}\sqrt{\frac{r}{a}}$$

$$\rho = \frac{2\sqrt{2ar}}{3}$$

$$\rho^2 = \frac{4(2ar)}{9}$$

$$\frac{\rho^2}{r} = \frac{8a}{9}$$

$\frac{\rho^2}{r}$ is constant.

Find ρ for the curve $r = a\sin(n\theta)$ at origin $(0,0)$

$$r_1 = \frac{dr}{d\theta} = a\cos(n\theta)n \text{ at } (0,0) = an\cos 0$$
$$= an$$

$$r_2 = \frac{d^2r}{d\theta^2} = -an^2\sin(n\theta) \text{ at } (0,0) = 0$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

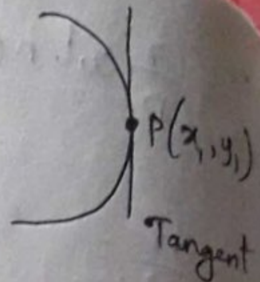
$$= \frac{(a^2 \sin^2(n\theta) + a^2 n^2)^{3/2}}{a^2 \sin^2 n\theta + 2a^2 n^2}$$

$$= \frac{(a^2 n^2)^{3/2}}{2a^2 n^2}$$

$$= \frac{a^3 n^3}{2a^2 n^2}$$

$$= \frac{an}{2}$$

Let $y = f(x)$ be the any curve



Slope of Tangent

$$\left(\frac{dy}{dx}\right)_{P(x_1, y_1)} = m$$

Equation of Tangent

$$\boxed{y - y_1 = m(x - x_1)}$$

The perpendicular distance from the point $P(x_1, y_1)$ to the line ~~is~~ $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

(*) Show that the radius of the curvature $x = a\cos^3\theta$ $y = a\sin^3\theta$ is equal to 3 times of the length of perpendicular from $(0,0)$ to the tangent.

Given,

$x = a\cos^3\theta$ $\qquad$ $y = a\sin^3\theta$

$x = a(\cos\theta)^3$ $\qquad$ $y = a(\sin\theta)^3$

D w.r.t. θ

$\frac{dx}{d\theta} = a\,3\cos^2\theta(-\sin\theta)$ $\qquad$ $\frac{dy}{d\theta} = a\,3\sin^2\theta(\cos\theta)$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$ $\qquad$ $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= -\frac{\sin\theta}{\cos\theta}$$

$$\frac{dy}{dx} = -\tan\theta$$

D w.r.t. x

$$\frac{d^2y}{dx^2} = -\sec^2\theta \cdot \frac{d\theta}{dx}$$

$$= -\sec^2\theta \cdot \frac{1}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{\sec^2\theta \cdot \sec^2\theta \cdot \operatorname{cosec}\theta}{3a}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^4\theta \cdot \operatorname{cosec}\theta}{3a}$$

radius of curvature $\Big\}$ $$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1 + (-\tan\theta)^2\right)^{3/2}}{\frac{\sec^4\theta \cdot \operatorname{cosec}\theta}{3a}}$$

$$= \frac{3a\left[1 + \tan^2\theta\right]^{3/2}}{\sec^4\theta \cdot \operatorname{cosec}\theta}$$

$$= \frac{3a\left[\sec^2\theta\right]^{3/2}}{\sec^4\theta \cdot \operatorname{cosec}\theta}$$

$$= \frac{3a(\sec^3\theta)}{\sec^4\theta \operatorname{cosec}\theta}$$

$$= \frac{3a}{\sec\theta \cdot \operatorname{cosec}\theta}$$

$$= 3a\cos\theta \cdot \sin\theta$$

$$\rho = 3a\sin\theta\cos\theta \quad \text{— (1)}$$

The equation of Tangent at the point $(\underset{y}{a\sin^3\theta}, \underset{x}{a\cos^3\theta})$

$(a\cos^3\theta, a\sin^3\theta)$

$$y - y_1 = m(x - x_1)$$

$$\cancel{y - a\cos^3\theta = m(x - a}$$

$$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$$

$$y - a\sin^3\theta = \frac{-\sin\theta}{\cos\theta}(x - a\cos^3\theta)$$

$$y\cos\theta - a\sin^3\theta \cdot \cos\theta = -x\sin\theta + a\cos^3\theta\sin\theta$$

$$x\sin\theta + y\cos\theta - a\sin^3\theta\cos\theta - a\cos^3\theta\sin\theta = 0$$

$$x\sin\theta + y\cos\theta - a\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta) = 0$$

$$x\sin\theta + y\cos\theta - a\sin\theta\cos\theta(1) = 0$$

The perpendicular from $(0,0)$ to the above line

$$d = \frac{|0+0-a\sin\theta\cos\theta|}{\sqrt{\sin^2\theta+\cos^2\theta}}$$

$$= \frac{a\sin\theta\cos\theta}{1}$$

$$d = a\sin\theta\cos\theta \quad \text{---} \quad (2)$$

from eq (1)

$$\rho = 3a\sin\theta\cos\theta$$

$$\rho = 3d$$

Hence proved //

Show that for the curve $x = a\cos\theta(1+\sin\theta)$ The radius of curvature is a $y = a\sin\theta(1+\cos\theta)$ the value of the parameter at the point for which the value of the parameter θ is $\frac{-\pi}{4}$

Given $x = a\cos\theta(1+\sin\theta)$

$$\frac{dx}{d\theta} = a\left[-\sin\theta(1+\sin\theta)+\cos\theta(\cos\theta)\right]$$

$$= a\left[-\sin\theta-\sin^2\theta+\cos^2\theta\right]$$

$$= a\left[\cos^2\theta-\sin^2\theta-\sin\theta\right]$$

$$\frac{dx}{d\theta} = a\left[\cos 2\theta-\sin\theta\right]$$

$$y = a\sin\theta(1+\cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\cos\theta(1+\cos\theta) + \sin\theta(-\sin\theta)\right]$$

$$= a\left[\cos\theta + \cos^2\theta - \sin^2\theta\right]$$

$$= a\left[\cos\theta + \cos 2\theta\right]$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\left[\cos\theta + \cos 2\theta\right]}{a\left[\cos 2\theta - \sin\theta\right]}$$

$$\left(\frac{dy}{dx}\right)_{\theta = \pi/4} = \frac{a\left[\cos(-\pi/4) + \cos(\pi/2)\right]}{a\left[\cos(\pi/2) - \sin(-\pi/4)\right]}$$

$$= \frac{\frac{1}{\sqrt{2}} + 0}{0 + 1/\sqrt{2}}$$

$$\left(\frac{dy}{dx}\right)_{\theta = \pi/4} = 1$$

$$\frac{dy}{dx} = \frac{a(\cos\theta + \cos 2\theta)}{a\left[\cos 2\theta - \sin\theta\right)}$$

D w.r.to y

$$\left(\frac{d^2y}{dx^2}\right)_{\theta = -\pi/4} = \frac{(\cos 2\theta - \sin\theta)(-2\sin 2\theta - \sin\theta)\cdot\frac{d\theta}{dx} - \left[\cos 2\theta + \cos\theta\right]\left(-2\sin 2\theta - \cos\theta\right)\cdot\frac{d\theta}{dx}}{(\cos 2\theta - \sin\theta)^2}$$

$$= \frac{\left(0+\frac{1}{\sqrt{2}}\right)\left(2+\frac{1}{\sqrt{2}}\right)\frac{\sqrt{2}}{a} - \left(0+\frac{1}{\sqrt{2}}\right)\left(2-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{2}}{a}}{1/2}$$

$$= 2\left[\frac{2\sqrt{2}+1-2\sqrt{2}+1}{\sqrt{2}a}\right]$$

$$= 2\left(\frac{2}{\sqrt{2}a}\right)$$

$$\Rightarrow \frac{4}{\sqrt{2}a} = \frac{4}{\sqrt{2}a}$$

$$\rho = \frac{[1+(1)]^{3/2}}{4/\sqrt{2}a}$$

$$= \frac{(2)^{3/2}}{4/\sqrt{2}a}$$

$$= \frac{\sqrt{8}\cdot\sqrt{2}a}{4}$$

$$\Rightarrow \frac{2\sqrt{2}\times\sqrt{2}a}{4} = a$$

(3) S.T the ρ at any point $x = ae^{\theta}(\sin\theta - \cos\theta)$, $y = ae^{\theta}(\sin\theta + \cos\theta)$ is twice the distance of tangent at the point from origin

$$x = ae^{\theta}(\sin\theta - \cos\theta)$$

$$\frac{dx}{d\theta} = a\left[e^{\theta}(\sin\theta - \cos\theta) + e^{\theta}(\cos\theta + \sin\theta)\right]$$

$$\frac{dx}{d\theta} = a\left[e^{\theta}(2\sin\theta)\right]$$

$$y = ae^{\theta}(\sin\theta + \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[e^{\theta}(\sin\theta + \cos\theta) + e^{\theta}(\cos\theta - \sin\theta)\right]$$

$$\frac{dy}{d\theta} = a\left[(2e^{\theta}\cos\theta)\right]$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta$$

$$\frac{dy}{dx} = \cot\theta$$

D w.r.t. θ

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2\theta \cdot \frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\theta \cdot \frac{1}{2ae^{\theta}\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-\operatorname{cosec}^3\theta}{2ae^{\theta}}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{(1 + \cot^2\theta)^{3/2}}{\frac{-\operatorname{cosec}^3\theta}{2ae^{\theta}}}$$

$\rho = -2ae^{\theta}$

$\rho = -2ae^{\theta}$

eqn of tangent

$y - y_1 = m(x - x_1)$

$y - ae^{\theta}(\sin\theta + \cos\theta)] = \cot\theta[x - ae^{\theta}(\sin\theta - \cos\theta)]$

$y\sin\theta - ae^{\theta}\sin\theta(\sin\theta + \cos\theta) = x\cos\theta + ae^{\theta}\cos^2\theta - ae^{\theta}\sin\theta\cos\theta$

$y\sin\theta - ae^{\theta}\sin^2\theta - ae^{\theta}\cos\theta\sin\theta = x\cos\theta + ae^{\theta}\cos^2\theta - ae^{\theta}\sin\theta\cos\theta$

$x\cos\theta - y\sin\theta + ae^{\theta}(1) = 0$

$\perp^{ar}$ distance from $(0,0)$ to the line

$$d = \frac{ae^{\theta}}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$d = ae^{\theta}$

$\rho = -2d$

$\rho = 2d$

(4) S.T the ρ of elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. at the end of the one of the major axis is equal to length of the semi lattus rectum.

Given that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

D w.r.t. x

$$\frac{2x}{a^2} + \frac{1}{b^2}\cdot 2y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{a^2}\cdot\frac{b^2}{2y}$$

$$\frac{dy}{dx} = \frac{-xb^2}{a^2y}$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{d}{dx}\left(\frac{x}{y}\right)\right]$$

$$= \frac{-b^2}{a^2}\left[\frac{y - x\cdot\frac{dy}{dx}}{y^2}\right]$$

$$= \frac{-b^2}{a^2}\left[\frac{y - x\left(\frac{-xb^2}{a^2y}\right)}{y^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{a^2y^2 + x^2b^2}{y^2}\right]$$

$$\rho = \frac{\left[1 + \frac{x^2b^4}{a^2y^2}\right]^{3/2}}{\frac{-b^2}{a^2}\left(\frac{a^2y^2 + x^2b^2}{a^2y^3}\right)}$$

$$\rho = \frac{\frac{-a^4y^3\left[a^4y^3 + x^2b^4\right]^{3/2}}{a^6y^3}}{b^2\left(a^2y^2 + x^2b^2\right)}$$

$$\rho(a,0) = \frac{\frac{a^4[a^4(0)+a^2b^4]^{3/2}}{a^6}}{b^2(0+a^2b^2)}$$

$$= \frac{\frac{-a^3b^6}{a^2}}{b^2(a^2b^2)}$$

$$= \frac{-a^3b^6}{a^4b^4}$$

$$\rho = \frac{-b^2}{a}$$

$$\rho = \frac{b^2}{a} \,//$$

*** imp

(6) P.T the ρ at any point of the curve $y = c\cosh\left(\frac{x}{c}\right)$ is varies the square of ordinate

$$y = c\cosh\left(\frac{x}{c}\right)$$

D. w.r.t x

$$\frac{dy}{dx} = c\sinh\left(\frac{x}{c}\right)\left(\frac{1}{c}\right)$$

$$\frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)$$

D w.r.t x

$$\frac{d^2y}{dx^2} = \cosh\left(\frac{x}{c}\right)\left(\frac{1}{c}\right)$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)}$$

$$= \frac{c\cdot\cosh^3(x/c)}{\cosh\left(\frac{x}{c}\right)}$$

$$= c\cosh^2\left(\frac{x}{c}\right)$$

$$\rho = c\cdot\left(\frac{y}{c}\right)^2$$

$$\rho = c\cdot\frac{y^2}{c^2}$$

$$\rho \propto y^2$$

Hence proved.

(*) find the ρ at $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point where the line $y=x$ cuts it.

eq of curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$

the above curve cut the line $y=x$

$$\sqrt{x}+\sqrt{x}=\sqrt{a}$$

$$2\sqrt{x}=\sqrt{a}$$

$$\sqrt{x}=\frac{\sqrt{a}}{2}$$

S.O.B.S $\quad x=\frac{a}{4}$

If $x = y$

$$y = \frac{a}{4}$$

The req point $\left(\frac{a}{4}, \frac{a}{4}\right)$

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

D. w.r.t x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\frac{1}{\sqrt{y}} \cdot \frac{dy}{dx} = \frac{-1}{\sqrt{x}}$$

$$\frac{dy}{dx} = \sqrt{\frac{y}{x}}$$

$$\left(\frac{dy}{dx}\right)_{\frac{a}{4}} = -1$$

$$\frac{d^2y}{dx^2} = \frac{-\left(\sqrt{x}\,\frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} - \sqrt{y} \cdot \frac{1}{2\sqrt{x}}\right)}{x}$$

$$\frac{d^2y}{dx^2} = \frac{-\left(\frac{-1}{2} - \frac{\sqrt{y}}{2\sqrt{x}}\right)}{x}$$

$$= \frac{-\left(\frac{-1}{2} - \frac{1}{2}\right)}{\frac{a}{4}}$$

$$= \frac{4}{a}$$

$$\rho = \frac{(2)^{3/2}}{4/a} \qquad \rho = \frac{a}{\sqrt{2}}$$

(*) find the point on the parabola $y^2 = 8x$ at which the ρ is $\frac{13}{16}$

given $y^2 = 8x$

$$2y \cdot \frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{4}{y}$$

$$\frac{d^2y}{dx^2} = 4\left(\frac{-1}{y^2}\right)\frac{dy}{dx}$$

$$= \frac{-4}{y^2} \cdot \left(\frac{4}{y}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-16}{y^3}$$

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\frac{125}{16} = \frac{\left(1 + \frac{16}{y^2}\right)^{3/2}}{-16/y^3}$$

$$\frac{125}{16} = \frac{-y^3\left(1 + \frac{16}{y^2}\right)^{3/2}}{-16}$$

$$125 = -y^3\left(1 + \frac{16}{y^2}\right)^{3/2}$$

$$125 = -\cancel{y^3}\frac{(y^2+16)^{3/2}}{\cancel{y^3}}$$

$$125 = -(y^2+16)^{3/2}$$

$$(125)^{2/3} = -(y^2+16)$$

$$5^2 = -(y^2+16)$$

$$25 = y^2+16$$

$$y^2 = 9$$

$$y = \pm 3$$

$$y^2 = 8x$$

$$9 = 8x$$

$$x = 9/8$$

points $\left(\frac{9}{8}, -3\right), \left(\frac{9}{8}, 3\right)$

(3) find ρ $x = a\left[\cos t + \log\left(\tan t/_2\right)\right]$. $y = a\sin t$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{\frac{1}{2}\sec^2 t/_2}{\tan t/_2}\right]$$

$$= a\left[-\sin t + \frac{1}{2}\,\frac{1}{\sin t/_2 \cos t/_2}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$\frac{dx}{dt} = a\left[-\sin t + \operatorname{cosec} t\right]$$

$$y = a\sin t$$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{a\cos t}{a\left[-\sin t + \frac{1}{\sin t}\right]}$$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} = \operatorname{Tan} t$$

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx}$$

$$= \sec^2 t \; \frac{1}{a\left[-\sin t + \frac{1}{\sin t}\right]}$$

$$= \frac{\sec^2 t \sin t}{a\cos^2 t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{a} \cdot \frac{\sin t}{\cos^4 t}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1 + \operatorname{Tan}^2(t)\right)^{3/2}}{\frac{\sin t}{a\cos^4 t}}$$

$$= \frac{a\sec^3 t \cos^4 t}{\sin t}$$

$\rho = a \cot t$

<u>Note</u>: If slope of the tangent is undefined $\left(\frac{dy}{dx}\right) = \infty$ then radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

(6) Prove that the radius of curvature at the point $(-2a, 2a)$ on the curve $x^2y = a(x^2+y^2)$ is $-2a$

The Given curve

$$x^2y = a(x^2+y^2)$$

d w.r.to x

$$x^2 \cdot \frac{dy}{dx} + y(2x) = a\left(2x + 2y \cdot \frac{dy}{dx}\right)$$

$$\Rightarrow x^2 \cdot \frac{dy}{dx} + 2xy = 2ax + 2ay \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x^2 - 2ay\right) = 2ax - 2xy$$

$$\frac{dy}{dx} = \frac{2ax - 2xy}{x^2 - 2ay}$$

At the point $(-2a, 2a)$

$$= \frac{2a(-2a) - 2(-2a)(2a)}{(-2a)^2 - 2a(2a)}$$

$$\Rightarrow \frac{-4a^2+8a^2}{4a^2-4a^2} = \frac{4a^2}{0} = \infty$$

Slope of tangent is undefined

Given that

$$x^2 y = a(x^2+y^2)$$

D w.r.t. y

$$x^2(1) + y(2x)\cdot\frac{dx}{dy} = a\left(2x\cdot\frac{dx}{dy} + 2y\right)$$

$$x^2 + 2xy\,\frac{dx}{dy} = 2ax\,\frac{dx}{dy} + 2ay$$

$$\frac{dx}{dy}\left[2xy - 2ax\right] = 2ay - x^2$$

$$\frac{dx}{dy} = \frac{2ay - x^2}{2xy - 2ax} \quad -①$$

$$= \frac{0}{2(-2a)(2a) - 2a(-2a)}$$

$$\frac{dx}{dy} = 0$$

$$\frac{d^2x}{dy^2} = \frac{(2xy-2ax)\left[2a(1) - 2x\,\frac{dx}{dy}\right] - (2ay - x^2)\left[2(x(1) + y\frac{dx}{dy}) - 2a\frac{dx}{dy}\right]}{(2xy-2ax)^2}$$

at the $(-2a, 2a)$

$$= \frac{2(-2a)(2a) - 2a(-2a)\left[2a - 2(-2a)(0)\right] - \left[2a(2a) - (-2a)^2\right]\left[2(-2a) + 2a - 0\right]}{\left[2(-2a)(2a) - 2a(-2a)\right]^2}$$

$$= \frac{\left[2(-2a)(2a) - 2a(-2a)\right]\left[2a - 0\right] - \left[2a(2a) - (-2a)^2\right]\left[2(-2a+0) - 0\right]}{\left[2(-2a)(2a) - 2a(-2a)\right]^2}$$

$$= \frac{(-8a^2 + 4a^2)(2a) - (4a^2 - 4a^2)(-4a)}{((-8a^2) + 4a^2)^2}$$

$$\frac{d^2x}{dy^2} = \frac{(-4a^2)(2a)}{(-4a^2)^2} = \frac{-8a^3}{16a^4} = \frac{-1}{2a}$$

Radius of Curvature

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{(1+0)^{3/2}}{\frac{-1}{2a}}$$

$$= \frac{1}{\frac{-1}{2a}}$$

$$\boxed{\rho = -2a} \quad //$$

imp (Lim) **

Newtonian Method

(i) If a curve passes through a origin and axis of x is the tangent, then the radius of curvature is

$$\rho = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^2}{2y}$$

(ii) If a curve passes through a origin and axis of y is the tangent, then the radius of curvature is

$$\rho = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{y^2}{2x}$$

(iii) The equation of tangents at the origin is obtained by equating the lowest degree term in the equation to 0

imp Q.V.

Example: find the radius of curvature at the origin to the curve $x^3-2x^2y+3xy^2-4y^3+5x^2-6xy+7y^2-8y=0$

Given curve,

$$x^3-2x^2y+3xy^2-4y^3+5x^2-6xy+7y^2-8y=0$$

The lowest degree term in the given equation $-8y$

The equation of the tangent at the origin $-8y=0$

$y=0$

The equation of the tangent at the origin is x-axis

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y}$$

Given curve,

$$x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0$$

Above eq^n divided with y.

$$\frac{x^3}{y} - \frac{2x^2y}{y} + \frac{3xy^2}{y} - \frac{4y^3}{y} + \frac{5x^2}{y} - \frac{6xy}{y} + \frac{7y^2}{y} - \frac{8y}{y} = 0$$

$$x \cdot \frac{x^2}{y} - 2x^2 + 3xy - 4y^2 + 5 \cdot \frac{x^2}{y} - 6x + 7y - 8 = 0$$

as $x \to 0$; $y \to 0$

$$0(2\rho) - 0 + 0 - 0 + 5(2\rho) - 0 + 0 - 8 = 0$$

$$10\rho - 8 = 0$$

$$10\rho = 8$$

$$\boxed{\rho = 4/5}$$

Example: Apply the Newtonian formula to find the radius of the curvature at the origin for the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

$$x = a(\theta + \sin\theta) \quad y = a(1 - \cos\theta)$$

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \quad \frac{dy}{d\theta} = a(0 + \sin\theta)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\frac{dy}{dx} = \mathrm{Tan}\,\theta/2$$

put $\theta = 0$

$$\frac{dy}{dx} = \mathrm{Tan}\,0/2$$

$$\frac{dy}{dx} = 0$$

$$\rho = \lim_{\theta\to 0} \frac{x^2}{2y}$$

$$= \lim_{\theta\to 0} \frac{a^2(\theta+\sin\theta)^2}{2a(1+\cos\theta)} = \frac{0}{0}$$

Apply L-Hospital rule

$$= \frac{a}{2} \lim_{\theta\to 0} \frac{2(\theta+\sin\theta)^{2-1}(1+\cos\theta)}{0+\sin\theta}$$

$$= \frac{2a}{2} \lim_{\theta\to 0} \frac{(\theta+\sin\theta)(1+\cos\theta)}{\sin\theta}$$

Again apply LH rule

$$= a \lim_{\theta\to 0} \frac{(\theta+\sin\theta)(0-\sin\theta)+(1+\cos\theta)(1+\cos\theta)}{\cos\theta}$$

$$= \frac{a\left(0+(1+1)(1+1)\right)}{1}$$

$$= a\,4$$

$$\boxed{\rho = 4a}$$

oTE: The radius of the curvature at the pole (origin) of the curve $r = f(\theta)$

$$\rho = \lim_{\theta \to 0} \frac{r}{2\theta}$$

xample: Find the radius of the curvature at the pole for the curve $r = a\sin(n\theta)$

Given $r = a\sin(n\theta)$

$$\rho = \lim_{\theta \to 0} \frac{r}{2\theta}$$

$$\left[\because \lim_{x \to 0} \frac{\sin ax}{x} = a\right]$$

$$= \lim_{\theta \to 0} \frac{a\sin(n\theta)}{2\theta}$$

$$= \frac{a}{2} \lim_{\theta \to 0} \frac{\sin(n\theta)}{\theta}$$

$$= \frac{a}{2} n$$

$$= \frac{na}{2}$$

(4) find ρ at the origin $(0,0)$ for the curve

$$x^3+y^3-2x^2+6y=0$$

Given eq^n $x^3+y^3-2x^2+6y=0$

The lowest degree term in the above eq^n is $6y$

The eq^n of tangent at the origin $6y=0$

$$y=0$$

The eq^n of tangent at the origin is x-axis

Radius of curvature $\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$

eq^n, $x^3+y^3-2x^2+6y=0$

D by y

$$x \cdot \frac{x^2}{y}+\frac{y^3}{y}-2\cdot\frac{x^2}{y}+\frac{6y}{y}=0$$

$$x(2\rho)+y^2-2(2\rho)+6=0$$

$$0+0-4\rho+6=0$$

$$4\rho=6$$

$$\rho=6/4$$

$$\boxed{\rho=3/2}$$

Find the radius of curvature at origin of the curve for the following.

(1) $2x^4+3y^4+4x^2y+xy-y^2+2x=0$

(2) $y=x^4-4x^3-18x^2$

(3) $3x^3+y^3+5y^2+3yx^2+2x=0$

(4) $x^4-y^4+x^3-y^3+x^2-y^2+y=0$

(5) find ρ for

$r=a(1-\cos\theta)$ at pole (origin)

Given $2x^4+3y^4+4x^2y+xy-y^2+2x=0$

The lowest degree term in the above eq^n $2x$

The eq^n of tangent at origin is $2x=0$

$\boxed{x=0}$

The eq^n of tangent at the origin is y axis

$$\rho=\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}$$

Given, eq^n ; $2x^4+3y^4+4x^2y+xy-y^2+2x=0$

D by x

$$2x^3+3\cdot\frac{y^2}{x}\cdot y^2+4xy+y-\frac{y^2}{x}+2=0$$

$$2(0)+3(2\rho)(0)+4(0)(0)+(0)-2\rho+2=0$$

$$-2\rho+2=0$$

$$2=2\rho$$

$$\boxed{\rho=1}$$

(2). Given $y=x^4-4x^3-18x^2$

$$x^4-4x^3-18x^2-y=0$$

The lowest degree term in the above eqⁿ is $-y=0$

The eqⁿ of tangent at origin is $y=0$

Eqⁿ of tangent at origin is x-axis

Radius of curvature $(\rho)=\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$

Given eqⁿ $x^4-4x^3-18x^2-y=0$

Divide by y

$$\frac{x^2}{y}\cdot x^2-4\frac{x^2}{y}\cdot x-18\frac{x^2}{y}-\frac{y}{y}=0$$

$$0-4(0)-18(2\rho)-1=0$$

$$-36\rho=1$$

$$\boxed{\rho=1/36}$$

[ρ can't be negative neglect the sign]

(3) Given $3x^3+y^3+5y^2+3yx^2+2x-6=0$

The lowest degree term in the above eqⁿ is $2x$

The eqⁿ of tangent at origin is $2x=0$

$x=0$

eqⁿ of tangent at origin is y-axis

Radius of curvature $(\rho) = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}$

Given eqⁿ is $3x^3+y^3+5y^2+3yx^2+2x-6=0$

Divide with x

$$3x^2+\frac{y^2}{x}\cdot y+5\left(\frac{y^2}{x}\right)+3xy+2-6=0$$

$$0+0+5(2\rho)+2\cdot 6=0$$

$$10\rho-4=0$$

$$\rho=\frac{4^2}{10_5}$$

$$\boxed{\rho=2/5}$$

(4) Given $x^4-y^4+x^3-y^3+x^2-y^2+y=0$

The lowest degree term in the above eqⁿ is y

The eqⁿ of tangent at origin is $y=0$

eqⁿ of tangent at origin is x axis

Radius of curvature $(\rho) = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$

Given eq^n is $x^4 - y^4 + x^3 - y^3 + x^2 - y^2 + y = 0$

Divide by y

$$\frac{x^2}{y} \cdot x^2 - y^3 + \frac{x^2}{y}(x) - y^2 + \frac{x^2}{y} - y + 1 = 0$$

$$2\rho + 1 = 0$$

$$2\rho = -1$$

$$\boxed{\rho = -\frac{1}{2}}$$

(x) Given $r = a(1-\cos\theta)$

$$\rho = \lim_{\theta \to 0} \frac{r}{2\theta}$$

$$= \lim_{\theta \to 0} \frac{a(1-\cos\theta)}{2\theta}$$

$$= \lim_{\theta \to 0} \frac{a\,\cancel{2}\sin^2\theta/2}{\cancel{2}\theta}$$

$$= \lim_{\theta \to 0} \frac{a\sin^2\theta/2}{\theta}$$

Coordinates of centre of curvature

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

[where $y_1 = \frac{dy}{dx}$

$y_2 = \frac{d^2y}{dx^2}$]

$$Y = y + \frac{1+y_1^2}{y_2}$$

centre of curvature (X, Y)

Circle of curvature :

The equation of circle at curvature is

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

where (X, Y) is centre of curvature

ρ = radius of curvature.

Example: Find the coordinates of centre of curvature

parabola $y^2 = 4ax$

Given equation

$$y^2 = 4ax \quad -(1)$$

D w.r.to x

$$2y \cdot \frac{dy}{dx} = 4a(1)$$

$$\frac{dy}{dx} = \frac{\overset{2}{\cancel{4}}a}{\cancel{2}y}$$

$$y_1 = \frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{d^2y}{dx^2} = 2a\frac{d}{dx}\left(\frac{1}{y}\right)$$

$$= 2a\left(\frac{-1}{y^2}\right)\cdot\frac{dy}{dx}$$

$$= \frac{-2a}{y^2}\cdot\frac{dy}{dx}$$

$$= \frac{-2a}{y^2}\left[\frac{2a}{y}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-4a^2}{y^3}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{\frac{2a}{y}\left(1+\left(\frac{2a}{y}\right)^2\right)}{\frac{-4a^2}{y^3}}$$

$$= x + \frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)\cdot\frac{y^3}{4a^2}$$

$$= x + \left(\frac{2a}{y}\right)\left(\frac{y^3}{4a^2}\right)\left(\frac{y^2+4a^2}{y^2}\right)$$

$$= x + \frac{\cancel{2}a}{\cancel{y}}\cdot\frac{y^{\cancel{3}}}{{}_2\cancel{4}a^2}\cdot\left(\frac{y^2+4a^2}{y^2}\right) \qquad (\because y^2 = 4ax)$$

$$= x + \frac{1}{2a}\left(4ax + 4a^2\right)$$

$= x + \frac{1}{2a}(4ax + 4a^2)$

$= x + \frac{1}{2a} 2a(2x + 2a)$

$= x + 2x + 2a$

$\boxed{X = 3x + 2a}$

$Y = y + \frac{(1 + y_1^2)}{y_2}$

$= y + \frac{1 + \left(\frac{2a}{y}\right)^2}{\frac{-4a^2}{y^3}}$

$= y - \frac{y^3}{4a^2}\left[\frac{y^2 + 4a^2}{y^2}\right]$ $\quad [\because y^2 = 4ax]$

$= y - \frac{y}{4a^2}\left[4ax + 4a^2\right]$

$= y - \frac{y}{4a^2} 4a(x + a)$

$= y - \frac{y}{a}(x + a)$

$= \frac{ay - xy - ay}{a}$

$\boxed{Y = \frac{-xy}{a}}$

$y^2 = 4ax$

$Y = \frac{-x\left[\pm\sqrt{4ax}\right]}{a}$

$y = \pm\sqrt{4ax}$

$$= \frac{\pm\, x\left[\frac{+}{-}\sqrt{4ax}\right]}{a}$$

$$= \mp \frac{x\, 2\sqrt{a}\sqrt{x}}{a}$$

$$= \mp \frac{2x\sqrt{a}\, x^{1/2}}{\sqrt{a}\,\sqrt{a}}$$

$$\boxed{Y = \mp \frac{2x^{3/2}}{(a)^{1/2}}}$$

$\therefore$ The centre of curvature (X, Y)

$$= \left(3x + a,\ \mp \frac{2x^{3/2}}{a^{1/2}}\right)$$

<u>Example</u> 2: Find centre of curvature of

$x = a\cos^3\theta \quad y = a\sin^3\theta \quad \left[\text{i.e } x^{2/3} + y^{2/3} = a^{2/3}\right]$

$x = a\cos^3\theta \qquad\qquad y = a\sin^3\theta$

$x = a(\cos\theta)^3 \qquad\qquad y = a(\sin\theta)^3$

$\frac{dx}{d\theta} = a\ 3(\cos\theta)^2(-\sin\theta) \qquad\qquad \frac{dy}{d\theta} = a\ 3\sin^2\theta\,(\cos\theta)$

$\frac{dx}{d\theta} = -3a\cos^2\theta\,\sin\theta \qquad\qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{-\sin\theta}{\cos\theta}.$$

$$y_1 = \frac{dy}{dx} = -\tan\theta$$

D w.r.t.o x

$$\frac{d^2y}{dx^2} = -\sec^2\theta \cdot \frac{d\theta}{dx}$$

$$= -\sec^2\theta \cdot \frac{1}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{\sec^2\theta \cdot \sec^2\theta \cdot \mathrm{cosec}\,\theta}{3a}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^4\theta \cdot \mathrm{cosec}\,\theta}{3a}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{(-\tan\theta)(1+\tan^2\theta)}{\frac{\sec^4\theta \cdot \mathrm{cosec}\,\theta}{3a}}$$

$$= x + \frac{\tan\theta(\sec^2\theta)}{\frac{\sec^4\theta \cdot \mathrm{cosec}\,\theta}{3a}}$$

$$= x + 3a \cdot \frac{\tan\theta}{\sec^2\theta\,\mathrm{cosec}\,\theta}$$

$= x + 3a\,\text{Tan}\theta \cos^2\theta \sin\theta$

$= x + 3a \dfrac{\sin\theta}{\cos\theta} \cos^2\theta \sin\theta$

$= x + 3a \sin^2\theta \cos\theta \qquad [\because x = a\cos^3\theta]$

$\boxed{X = a\cos^3\theta + 3a\sin^2\theta \cos\theta}$

$Y = y + \dfrac{1 + y_1^2}{y_2}$

$Y = y + \dfrac{(1 + (-\text{Tan}\theta)^2)}{\dfrac{\sec^4\theta \cdot \text{cosec}\theta}{3a}}$

$= y + \dfrac{(1 + \text{Tan}^2\theta)\, 3a}{\sec^4\theta \,\text{cosec}\theta}$

$= y + \dfrac{\sec^2\theta\, 3a}{\sec^4\theta\, \text{cosec}\theta} \qquad [\because y = a\sin^3\theta]$

$= a\sin^3\theta + \dfrac{3a}{\sec^2\theta\, \text{cosec}\theta}$

$\boxed{Y = a\sin^3\theta + 3a\cos^2\theta \sin\theta}$

$\therefore$ The centre of curvature (X, Y)

$\Rightarrow \left(a\cos^3\theta + 3a\sin^2\theta\cos\theta,\ a\sin^3\theta + 3a\cos^2\theta\sin\theta\right)$

Q) find circle of curvature at point (0,0) for the curve $x+y = x^2+y^2+x^3$

G. $x+y = x^2+y^2+x^3$

D w.r.to x

$$1+\frac{dy}{dx} = 2x+2y\cdot\frac{dy}{dx}+3x^2$$

$$\frac{dy}{dx}-2y\cdot\frac{dy}{dx} = 3x^2+2x-1$$

$$\frac{dy}{dx}(1-2y) = 3x^2+2x-1$$

$$\frac{dy}{dx} = \frac{3x^2+2x-1}{1-2y}$$

$$\left(\frac{dy}{dx}\right)_{(0,0)} = \frac{-1}{1}$$

$$\left(\frac{dy}{dx}\right)_{(0,0)} = -1$$

D w.r.to x

$$\frac{d^2y}{dx^2} = \frac{(1-2y)(6x+2)-(3x^2+2x-1)\left(1-2\cdot\frac{dy}{dx}\right)}{(1-2y)^2}$$

$$= \frac{(1-2(0))(6(0)+2)-(3(0)+2(0)-1)(1-2(-1))}{(1-2(0))^2}$$

$$\Rightarrow \frac{1(2)+2}{1} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{(0,0)} = 4$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \qquad Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= 0 - \frac{(-1)(1+1)}{4} \qquad = 0 + \frac{1+1}{4}$$

$$\boxed{X = 1/2} \qquad \boxed{Y = 1/2}$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$= \frac{(1+1)^{3/2}}{4}$$

$$= \frac{2\sqrt{2}}{4}$$

$$\boxed{\rho = \frac{1}{\sqrt{2}}}$$

eqn of circle is $(x-X)^2 + (y-Y)^2 = \rho^2$

$$\left(x-\tfrac{1}{2}\right)^2 + \left(y-\tfrac{1}{2}\right)^2 = \frac{1}{2}$$

$$\frac{(2x-1)^2 + (2y-1)^2}{4} = \frac{1}{2}$$

$$4x^2 + 1 - 4x + 4y^2 + 1 - 4y = 2$$

$$4x^2 + 4y^2 - 4x - 4y + 2 - 2 = 0$$

$$x^2 + y^2 - x - y = 0 \,//$$

1) Find ρ and centre of curvature at $(+1, -1)$ for the curve $y = x^3 - 6x^2 + 3x + 1$

$$y = x^3 - 6x^2 + 3x + 1$$

D w.r.to x

$$\frac{dy}{dx} = 3x^2 - 12x + 3$$

$$\left(\frac{dy}{dx}\right)_{(1,-1)} = 3(+1)^2 - 12(1) + 3$$

$$= 3 - 12 + 3$$

$$= -6$$

D w.r.t x

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1,-1)} = -6$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 1 + \frac{6(1+36)}{-6}$$

$$= 1 - 37$$

$$\boxed{X = -36}$$

$$Y = y + \frac{(1+y_1)^2}{y_2}$$

$$= -1 + \frac{(1+36)}{-6}$$

$$= -1 - \frac{37}{6}$$

$$Y = \frac{-6-37}{6} \Rightarrow \frac{-43}{6}$$

$$\boxed{Y = -43/6}$$

* imp

(*) find the circle of curvature $x = a(\cos t + t\sin t)$

$y = a(\sin t - t\cos t)$ at $t = \pi/4$

Given $x = a(\cos t + t\sin t)$ $\quad y = a(\sin t - t\cos t)$

D wrt t

$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$ $\quad \frac{dy}{dx} = a(\cos t - (t(-\sin t) + \cos t))$

$= at\cos t$ $\quad = a(\cos t + t\sin t - \cos t)$

$= at\sin t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = a\tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at\cos t} = \frac{\sec^3 t}{at}$$

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1 + \tan^2 t)^{3/2}}{\frac{\sec^3 t}{at}}$$

$$= \frac{at(\sec^2 t)^{3/2}}{\sec^3 t} = \frac{at\sec^3 t}{\sec^3 t}$$

$$= at$$

eq^n of curvature $(x - X)^2 + (y - Y)^2 = \rho^2$

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \qquad Y = x + \frac{(1+y_1^2)}{y_2}$$

$$= x - \frac{\tan t\,(1+\tan^2 t)}{\frac{\sec^2 t}{at}} \qquad = y + \frac{(1+\tan^2 t)\,at}{\sec^3 t}$$

$$= x - \frac{\tan t\,(\sec^2 t)\,at}{\sec^3 t} \qquad = y + \frac{(\sec^2 t)\,at}{\sec^3 t}$$

$$= a\sin t - at\cos t + at\cos t$$

$$= x - \frac{at\tan t}{\sec t} \qquad = a\sin t$$

$$Y = \frac{a}{\sqrt{2}}$$

$$= a\cos t + at\cancel{\sin t} - \frac{\sin t}{\cos t}\,at\cdot\cos t$$

$$X = a\cos t \qquad [t = \pi/4]$$

$$a = \frac{a}{\sqrt{2}}$$

eq^n of curvature is

$$(x - X)^2 + (y - Y)^2 = \rho^2$$

$$\left(x - \frac{a}{\sqrt{2}}\right)^2 + \left(y - \frac{a}{\sqrt{2}}\right)^2 = a^2t^2$$

$$(x - a\cos t)^2 + (y - a\sin t)^2 = a^2t^2$$

$$x^2 + a^2\cos^2 t - 2ax\cos t + y^2 + a^2\sin^2 t - 2ay\sin t = a^2t^2$$

$$x^2 + y^2 - 2a(x\cos t + y\sin t) + a^2 = a^2t^2$$

$$x^2+y^2-2a(x\cos t+y\sin t)+a^2(1-t^2)=0$$

(8) find eq^n of circle of curvature at point $\left(\frac{a}{4},\frac{a}{4}\right)$, of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

D w.r.to x

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{-\sqrt{\frac{a}{4}}}{\sqrt{\frac{a}{4}}}$$

$$=-1$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=-1$$

D w.r.to x

$$\frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

$$\frac{d^2y}{dx^2}=\frac{-\left(\sqrt{x}\,\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}\right)}{x}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{-\left[\sqrt{\frac{a}{4}}\cdot\frac{1}{2\sqrt{\frac{a}{4}}}(-1)-\sqrt{\frac{a}{4}}\cdot\frac{1}{2\sqrt{\frac{a}{4}}}\right]}{a/4}$$

$$\frac{d^2y}{dx^2} = \frac{-\left(\frac{-1}{2} \cdot \frac{1}{2}\right)}{\frac{a}{4}}$$

$$= \frac{-4(-1)}{a}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{a}{4}, \frac{a}{4}\right)} = \frac{4}{a}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= \frac{a}{4} - \frac{(-1)(1+1)a}{4}$$

$$= \frac{a+2a}{4}$$

$$X = \frac{3a}{4}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= \frac{a}{4} + \frac{(1+1)a}{4}$$

$$= \frac{a+2a}{4}$$

$$= \frac{3a}{4}$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$= \frac{(1+1)^{3/2}}{4} \cdot a$$

$$= \frac{2\sqrt{2}}{4} \cdot a$$

$$\rho = \frac{a}{\sqrt{2}}$$

Eq^n of circle of curvature is

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

$$\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$$

$$\frac{(4x-3a)^2+(4y-3a)^2}{16}=\frac{a^2}{2}$$

$$16x^2+9a^2-24a+16y^2+9a^2-24ay=8a^2$$

$$16x^2+18a^2-24a(x+y)+16y^2=8a^2$$

$$8x^2+9a^2-12a(x+y)+8y^2-4a^2=0$$

$$8(x^2+y^2)-12a(x+y)+5a^2=0$$

D.D

(4) find the eqn of circle of the curvature of the curve $y^2=4ax$

$$y^2=4ax$$

D wrto x

$$2y\cdot\frac{dy}{dx}=4a$$

$$\frac{dy}{dx}=\frac{2a}{y}$$

$$\frac{d^2y}{dx^2}=2a\left(\frac{-1}{y^2}\right)\cdot\frac{dy}{dx}$$

$$=2a\left(-\frac{1}{y^2}\right)\left(\frac{2a}{y}\right)$$

$$\frac{d^2y}{dx^2}=\frac{-4a^2}{y^3}$$

$$X=x-\frac{\frac{2a}{y}\left(1+\frac{2a^2}{y}\right)y^3}{-4a^2}$$

$$Y=y+\frac{(1+y_1^2)}{y_2}$$

$$=\frac{-2x^{3/2}}{a^{1/2}}$$

$$= x + \frac{2a}{4a^2} y^2 \left(\frac{y^2+4a^2}{y^2}\right) \qquad Y = \frac{-2x^{3/2}}{a^{1/2}}$$

$$X = 3x + 2a$$

$$= \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left(1+\frac{4a^2}{y^2}\right)^{3/2} y^3}{4a^2}$$

$$= \frac{-(y^2+4a^2)^{3/2} y^3}{y^3 4a^2}$$

$$= \frac{-(y^2+4a^2)^{3/2}}{4a^2}$$

$$\rho^2 = \frac{(y^2+4a^2)^{3/2}}{16a^4}$$

∴ eq circle $(x-X)^2 + (y-Y)^2 = \rho^2$

$$(x-3x-2a)^2 + \left(y + \frac{2x^{3/2}}{a^{1/2}}\right)^2 = \left(\frac{y^2+4a^2}{16a^4}\right)^3$$

$$(-2x-2a)^2 + \left(\frac{a^{1/2}y + 2x^{3/2}}{a^{1/2}}\right)^2 = \left(\frac{y^2+4a^2}{16a^4}\right)^3$$

$$4(x+a)^2 + \left(\frac{a^{1/2}y + 2x^{3/2}}{a^{1/2}}\right)^2 = \frac{(y^2+4a^2)^3}{16a^4}$$

(4) find the circle of curvature of $ay^2 = x^3$ at $P(a,a)$

eqⁿ of the given curve is $ay^2 = x^3$

D wrt x

$$a(2y)\cdot\frac{dy}{dx} = 3x^2$$

D wrt x

$$a\left[2y\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}(2)\cdot\frac{dy}{dx}\right] = 3(2x)$$

$$2a\left[y\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)\right] = 6x$$

$$y\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) = \frac{3x}{a}$$

$$\frac{d^2y}{dx^2} = \frac{1}{y}\left[\frac{3x}{a} - \left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{y}\left[\frac{3x}{a} - \left(\frac{3x^2}{2ay}\right)\left(\frac{3x^2}{2ay}\right)\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{y}\left[\frac{3x}{a} - \frac{9x^4}{4a^2y^2}\right]$$

At point (a,a)

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\left(\frac{dy}{dx}\right)_{a,a} = \frac{3a^2}{2a^2} = 3/2$$

$$\left(\frac{d^2y}{dx^2}\right)_{a,a} = \frac{1}{a}\left[\frac{3a}{2a} - \frac{9a^4}{4a^4}\right]$$

$$= \frac{1}{a}\left[\frac{12-9}{4}\right]$$

$$= \frac{1}{a}\left[\frac{3}{4}\right]$$

$$= \frac{3}{4a}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a - \frac{\frac{3}{2}\left(1+\frac{9}{4}\right]}{3/4a}$$

$$= a - \frac{3}{2}\cdot\frac{4a}{3}\cdot\frac{13}{4}$$

$$= a - \frac{13a}{2}$$

$$= \frac{2a-13a}{2}$$

$$X = \frac{-11a}{2}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= \frac{a + \left(1+\frac{9}{4}\right)4a}{3}$$

$$= \frac{a + (4+9)4a}{4\times 3}$$

$$= \frac{3a+13a}{3}$$

$$= \frac{16a}{3}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left(1+\frac{9}{4}\right)^{3/2}}{3/4a} = \frac{4a}{3}\cdot\left(\frac{13}{4}\right)^{3/2} = \frac{4a}{\sqrt{3}}\cdot\frac{4a}{3}\sqrt{\left(\frac{13}{4}\right)^3}$$

$$= \frac{4a}{3}\sqrt{\frac{13\times 13\times 13}{64}}$$

$$= \frac{4a}{3}\,\frac{13\sqrt{3}}{8}$$

$$\rho = \frac{a\,13\sqrt{3}}{6}$$

eqn of circle of curvature;

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

$$\left(x-\left(\frac{-11a}{2}\right)\right)^2 + \left(y-\frac{16a}{3}\right) = \left(\frac{13a\sqrt{3}}{6}\right)^2$$

$$\left(\frac{2x+11a}{2}\right)^2 + \left(\frac{3y-16a}{3}\right)^2 = \frac{a^2(3)^3}{36} \quad //$$

Evolutes. (20m) v.*imp

We know that the coordinates of centre of curvature are given (X, Y)

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \qquad Y = y + \frac{(1+y_1^2)}{y_2}.$$

Now we eliminate x and y [Parameters θ (or) t)

Then the relation between X and Y is the required equation of the evolutes

Example 1:- Obtain the evolute of parabola $y^2 = 4ax$

Given equation $y^2 = 4ax$ —(1)

D w.r.to ~~x~~

$$2y \cdot \frac{dy}{dx} = 4a(1)$$

$$\frac{dy}{dx} = \frac{\overset{2}{\cancel{4}}a}{\cancel{2}y}$$

$$y_1 = \frac{dy}{dx} = \frac{2a}{y}$$

Again d w.r.to x

$$\frac{d^2y}{dx^2} = 2a \frac{d}{dx}\left(\frac{1}{y}\right)$$

$$= 2a\left(\frac{-1}{y^2}\right) \cdot \frac{dy}{dx}$$

$$= \frac{-2a}{y^2} \cdot \frac{dy}{dx}$$

$$= \frac{-2a}{y^2}\left(\frac{2a}{y}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-4a^2}{y^3}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{\frac{2a}{y}\left(1+\left(\frac{2a}{y}\right)^2\right)}{\frac{-4a^2}{y^3}}$$

$$= x + \frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)\frac{y^3}{4a^2}$$

$$= x + \left(\frac{2a}{y}\right)\left(\frac{y^3}{4a^2}\right)\left(\frac{y^2+4a^2}{y^2}\right)$$

$$= x + \frac{\cancel{2}a}{\cancel{y}} \cdot \frac{y^{\cancel{3}}}{\cancel{4}_{2}a^2} \cdot \frac{y^2+4a^2}{y^2}$$

$$= x + \frac{1}{2a}(4ax+4a^2)$$

$$= x + \frac{1}{\cancel{2a}}(\cancel{2a})(2x+2a)$$

$$= x + 2x + 2a$$

$$\boxed{X = 3x+2a}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= y + \frac{\left(1+\left(\frac{2a}{y}\right)^2\right)}{\frac{-4a^2}{y^3}}$$

$$= y - \frac{y^3}{4a^2}\left(\frac{y^2-4a^2}{y^2}\right)$$

$$= y - \frac{y}{4a^2}(4ax+4a)$$

$$= y - \frac{y}{4a^2}4a(x+a)$$

$$= y - \frac{y}{a}(x+a)$$

$$= \frac{ay-xy-ay}{a}$$

$$\boxed{Y = \frac{-xy}{a}}$$

$$Y = \frac{-x\left[\pm\sqrt{4ax}\right]}{a}$$

$$Y = \mp \frac{x\,2\sqrt{a}\sqrt{x}}{a}$$

$$Y = \mp \frac{2x\sqrt{a}x^{1/2}}{\sqrt{a}\sqrt{a}}$$

$$\boxed{Y = \mp \frac{2x^{3/2}}{(a)^{1/2}}}$$

$\Rightarrow X = 3x + 2a$

$3x = X - 2a$

$x = \frac{X - 2a}{3}$

sub in Y

$$Y = \mp \frac{2x^{3/2}}{a^{1/2}}$$

$$Y = \mp \frac{2\left[\frac{X-2a}{3}\right]^{3/2}}{a^{1/2}}$$

S.O.B.S

$$Y^2 = \frac{4\left(\frac{X-2a}{3}\right)^3}{a}$$

$$aY^2 = \frac{4(X-2a)^3}{27}$$

$$27aY^2 = 4(X-2a)^3$$

The required evolute eq^n is

$27ay^2 = 4(x-2a)^3$ //

Ex: Find evolute of astroid

$x = a\cos^3\theta$, $y = a\sin^3\theta$. (or) $x^{2/3} + y^{2/3} = a^{2/3}$

$x = a\cos^3\theta$

D w.r.to θ

$$\frac{dx}{d\theta} = a\,3\cos^2\theta(-\sin\theta)$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$y = a\sin^3\theta$$

d w.r.t. θ

$$\frac{dy}{d\theta} = a\,3\sin^3\theta\cos\theta$$

$$\frac{dy}{d\theta} = 3a\sin^3\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$\frac{dy}{dx} = -\tan\theta$$

D w.r.t. x

$$\frac{d^2y}{dx^2} = -\sec^2\theta\cdot\frac{d\theta}{dx}$$

$$= -\sec^2\theta\cdot\frac{1}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{\sec^2\theta\cdot\sec^2\theta\,\mathrm{cosec}\,\theta}{3a}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^4\theta\cdot\mathrm{cosec}\,\theta}{3a}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{(-\tan\theta)(1+(-\tan\theta)^2)}{\frac{\sec^4\theta\,\mathrm{cosec}\,\theta}{3a}}$$

$$= x + \frac{\tan\theta \sec^2\theta}{\frac{\sec^4\theta \operatorname{cosec}\theta}{3a}}$$

$$= x + 3a \frac{\tan\theta}{\sec^2\theta \operatorname{cosec}\theta}$$

$$= x + 3a \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \cos^{\cancel{2}}\theta \cdot \sin\theta$$

$$X = x + 3a\sin^2\theta\cos\theta$$

$$X = a\cos^3\theta + 3a\sin^2\theta\cos\theta$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= y + \frac{(1+(-\tan\theta)^2)}{\frac{\sec^4\theta \cdot \operatorname{cosec}\theta}{3a}}$$

$$= y + \frac{(1+\tan^2\theta)}{\frac{\sec^4\theta \cdot \operatorname{cosec}\theta}{3a}}$$

$$= y + \frac{3a\sec^2\theta}{\sec^4\theta \cdot \operatorname{cosec}\theta}$$

$$= y + 3a\cos^2\theta \cdot \sin\theta$$

$$= a\sin^3\theta + 3a\cos^2\theta\sin\theta$$

$$X + Y = a\cos^3\theta + a\sin^3\theta + 3a\sin^2\theta\cos\theta + 3a\cos^2\theta\sin\theta$$

$$X + Y = a\left[\cos^3\theta + \sin^3\theta + 3\cos\theta\sin\theta(\sin\theta + \cos\theta)\right]$$

$$(\because a^3+b^3+3ab(a+b) = (a+b)^3$$

$$x+y = a[\cos\theta+\sin\theta]^3$$

$$(x+y)^{1/3} = a^{1/3}[\cos\theta+\sin\theta]$$

squaring on bs

$$(x+y)^{2/3} = a^{2/3}(\cos\theta+\sin\theta)^2 \text{ --- } (1)$$

$$(x-y)^{2/3} = a^{2/3}(\cos\theta-\sin\theta)^2 \text{ --- } (2)$$

eq (1) + eq (2)

$$(x+y)^{2/3}+(x-y)^{2/3} = a^{2/3}(\cos\theta+\sin\theta)^2 + a^{2/3}(\cos\theta-\sin\theta)^2$$

$$= a^{2/3}[\cos^2\theta+\sin^2\theta+2\cos\theta\sin\theta+\cos^2\theta+\sin^2\theta - 2\cos\theta\sin\theta]$$

$$= a^{2/3}[2\cos^2\theta+2\sin^2\theta]$$

$$= 2a^{2/3}[\cos^2\theta+\sin^2\theta]$$

$$= 2a^{2/3}(1)$$

$$(x+y)^{2/3}+(x-y)^{2/3} = 2a^{2/3}$$

The required evolute

$$(x+y)^{2/3}+(x-y)^{2/3} = 2a^{2/3}$$

3) $x = a\left(\cos t+\log\left(\tan\frac{t}{2}\right)\right]$, $y = a\sin t$.

Given, $x = a\left(\cos t+\log\left(\tan\frac{t}{2}\right)\right)$, $y = a\sin t$

D w.r.t. t

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}} \sec^2\left(\frac{t}{2}\right)\frac{1}{2}\right)$$

$$= a\left(-\sin t + \cot\left(\frac{t}{2}\right)\frac{1}{\cos^2\left(\frac{t}{2}\right)} \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{\cos t/2}{\sin t/2} \cdot \frac{1}{\cos^2 t/2} \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) \qquad \left(\because 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left[\frac{-\sin^2 t + 1}{\sin t}\right]$$

$$\frac{dx}{dt} = a\frac{\cos^2 t}{\sin t}$$

GT $y = a\sin t$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}}$$

$$= \frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} = \operatorname{Tan} t$$

D w.r.t. x

$$\frac{d^2y}{dx^2} = \sec^2 t \frac{dt}{dx}$$

$$= \sec^2 t \cdot \frac{\sin t}{a\cos^2 t}$$

$$= \frac{\sec^2 t \cdot \sin t \cdot \sec^2 t}{a}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^4 t \cdot \sin t}{a}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}}$$

$$= \frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} = \operatorname{Tan} t$$

D w.r.t. x

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx}$$

$$X = x - \frac{y(1+y_1^2)}{y_2}$$

$$= x - \frac{\operatorname{Tan} t\,(1+\operatorname{Tan}^2 t)}{\sec^4 t \sin t / a}$$

$$= x - \frac{a \tan t \sec^2 t}{\sec^2 t \sin t}$$

$$= x - a \frac{\sin t}{\cos t} \cdot \frac{\cos^2 t}{\sin t}$$

$$= x - a \cos t$$

$$= a\left[\cos t + \log\left(\tan t/2\right)\right] - a\cos t$$

$$= a\cos t + a\log\left(\tan t/2\right) - a\cos t$$

$$X = a\log\left(\tan t/2\right)$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= y + \frac{(1+\tan^2 t)}{\frac{\sec^4 t \sin t}{a}}$$

$$= y + a \cdot \frac{\sec^2 t}{\sec^4 t \sin t}$$

$$= y + a \frac{1}{\sec^2 t \sin t}$$

$$= a \sin t + \frac{a\cos^2 t}{\sin t}$$

$$= \frac{a(\sin^2 t + \cos^2 t)}{\sin t}$$

$$Y = \frac{a(1)}{\sin t} = \frac{a}{\sin t}$$

$x = a\log(\mathrm{Tan}\, t/2)$

$\frac{x}{a} = \log(\tan t/2)$

$\mathrm{Tan}\, t/2 = e^{x/a}$

$a = \log N$

$N = e^a$

$y = \frac{a}{\sin t}$ $\left(\because \sin 2\theta = \frac{2\,\mathrm{Tan}\,\theta}{1+\mathrm{Tan}^2\theta}\right)$

$y = \frac{a}{\frac{2\,\mathrm{Tan}\, t/2}{1+\mathrm{Tan}^2 t/2}}$

$= \frac{a(1+\mathrm{Tan}^2 t/2)}{2\tan t/2}$

$= \frac{a(1+(e^{x/a})^2)}{2e^{x/a}}$

$= \frac{a}{2}\left[\frac{1+e^{2x/a}}{e^{x/a}}\right]$

$= \frac{a}{2}\left[\frac{1}{e^{x/a}} + \frac{e^{2x/a}}{e^{x/a}}\right]$

$= a\left[\frac{e^{-x/a}+e^{x/a}}{2}\right]$

$y = a\cosh\left(\frac{x}{a}\right)$

The evolute is

$$y = a\cosh(x/a)$$

Hence proved

Q find the evolue of the curve $x = a(\cos\theta + \theta\sin\theta)$

$y = a[\sin\theta - \theta\cos\theta)$

Given $x = a(\cos\theta + \theta\sin\theta)$

$$\frac{dx}{d\theta} = a[-\sin\theta + \theta\cos\theta + \sin\theta \cdot 1]$$

$$\frac{dx}{d\theta} = a(\theta\cos\theta)$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\frac{dy}{d\theta} = a(\sin\theta - (\theta(-\sin\theta) + \cos\theta \cdot 1]$$

$$= a(\cos\theta + \theta\sin\theta - \cos\theta)$$

$$\frac{dy}{d\theta} = a(\theta\sin\theta)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta}$$

$$\frac{dy}{dx} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \sec^2\theta \cdot \frac{d\theta}{dx}$$

$$= \sec^2\theta \cdot \frac{1}{a\theta\cos\theta}$$

$$= \frac{\sec^2\theta \sec\theta}{a\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a(\cos\theta + \theta\sin\theta) - \frac{\tan\theta(1+\tan^2\theta)}{\frac{\sec^3\theta}{a\theta}}$$

$$= a\cos\theta + a\theta\sin\theta - \frac{\tan\theta \sec^2\theta}{\sec^3\theta} \cdot a\theta$$

$$= a\cos\theta + a\theta\sin\theta - \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\sec\theta} \cdot a\theta$$

$$= a\cos\theta + a\theta\sin\theta - a\theta\sin\theta$$

$$\boxed{X = a\cos\theta}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$Y = a(\sin\theta - \theta\cos\theta) + \frac{(1+\tan^2\theta)}{\frac{\sec^3\theta}{a\theta}}$$

$$Y = a\sin\theta - a\theta\cos\theta + \frac{\sec^2\theta}{\sec^3\theta} \cdot a\theta$$

$$Y = a\sin\theta - a\theta\cos\theta + a\theta\cos\theta$$

$$Y = a\sin\theta$$

$X = a\cos\theta$ $\qquad$ $Y = a\sin\theta$

$X/a = \cos\theta$ $\qquad$ $Y/a = \sin\theta$

$\left(\frac{x}{a}\right)^2 = \cos^2\theta$ $\qquad\qquad$ $\left(\frac{y}{a}\right)^2 = \sin^2\theta$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = \cos^2\theta + \sin^2\theta$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$x^2 + y^2 = a^2$$

Required evolute is

$$x^2 + y^2 = a^2 //$$

ST evolute of the elipse

$x = a\cos\theta \quad y = b\sin\theta$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

$x = a\cos\theta \qquad\qquad y = b\sin\theta$

$\frac{dx}{d\theta} = a(-\sin\theta) \qquad \frac{dy}{d\theta} = b\cos\theta$

$= -a\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta}$$

$$\frac{dy}{dx} = \frac{-b}{a}\cot\theta$$

D wrto x

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\left(-\operatorname{cosec}^2\theta\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{b}{a} \operatorname{cosec}^2\theta \cdot \frac{1}{-a\sin\theta}$$

$$= \frac{-b}{a^2} \operatorname{cosec}^2\theta \cdot \operatorname{cosec}\theta$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2} \operatorname{cosec}^3\theta$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$X = x - \frac{\left(-\frac{b}{a}\right)\cot\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\frac{-b}{a^2}\operatorname{cose}^3\theta}$$

$$= x - \frac{\cancel{b}}{a}\frac{a^{\cancel{2}}}{\cancel{b}} \cot\theta \sin^3\theta \left(\frac{a^2+b^2\cot^2\theta}{a^2}\right)$$

$$x - \frac{1}{a}\frac{\cos\theta}{\sin\theta} \sin^3\theta\left(a^2+b^2\cot^2\theta\right)$$

$$= x - \frac{1}{a}\cos\theta \sin^2\theta \cdot \left(a^2+b^2\cot^2\theta\right)$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\sin^2\theta\left(a^2+b^2\cot^2\theta\right)$$

$$= a\cos\theta - \frac{\cos\theta\sin^2\theta\left(a^2+b^2\cot^2\theta\right)}{a}$$

$$= \frac{a^2\cos\theta - \cos\theta\sin^2\theta\left(a^2+b^2\cot^2\theta\right)}{a}$$

$$= \frac{a^2\cos\theta - a^2\cos\theta\sin^2\theta - b^2\cos\theta\sin^2\theta\cot^2\theta}{a}$$

$$= \frac{a^2\cos\theta(1-\sin^2\theta) - b^2\cos\theta\sin^2\theta\,\frac{\cos^2\theta}{\sin^2\theta}}{a}$$

$$= \frac{a^2\cos\theta(\cos^2\theta) - b^2\cos\theta\cos^2\theta}{a}$$

$$x = \frac{\cos^3\theta(a^2-b^2)}{a}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= y + \frac{\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\frac{-b}{a^2}\operatorname{cosec}^3\theta}$$

$$= y - \frac{a^2}{b}\sin^3\theta\left(\frac{a^2+b^2\cot^2\theta}{a^2}\right)$$

$$= b\sin\theta - \frac{\sin^3\theta(a^2+b^2\cot^2\theta)}{b}$$

$$= \frac{b^2\sin\theta - a^2\sin^3\theta - b^2\sin^3\theta\cot^2\theta}{b}$$

$$= \frac{b^2\sin\theta - a^2\sin^3\theta - b^2\sin^3\theta\cdot\frac{\cos^2\theta}{\sin^2\theta}}{b}$$

$$= \frac{b^2\sin\theta - a^2\sin^3\theta - b^2\sin\theta\cos^2\theta}{b}$$

$$= \frac{b^2\sin\theta(1-\cos^2\theta) - a^2\sin^3\theta}{b}$$

$$y = \frac{\sin^3\theta\,(b^2-a^2)}{b}$$

$$y = \frac{-(a^2-b^2)\sin^3\theta}{b}$$

$$x = \frac{(a^2-b^2)\cos^3\theta}{a} \qquad y = \frac{-(a^2-b^2)\sin^3\theta}{b}$$

$$ax = (a^2-b^2)\cos^3\theta \qquad (by) = -(a^2-b^2)\sin^3\theta$$

$$(ax)^{2/3} = (a^2-b^2)^{2/3}(\cos^3\theta)^{2/3} \qquad (by)^{2/3} = (-(a^2-b^2))^{2/3}(\sin^3\theta)^{2/3}$$

$$(ax)^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta \quad \text{--- (1)} \qquad (by)^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta \quad \text{--- (2)}$$

eq (1) + eq (2)

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta + (a^2-b^2)^{2/3}\sin^2\theta$$

$$= (a^2-b^2)^{2/3}(\cos^2\theta + \sin^2\theta)$$

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}(1)$$

$\therefore$ The evolute eq^n is

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3} \; //$$

Hence proved.

(4) S.T For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the eq^n of evolute is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$

Given eq^n of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

A point on hyperbola is $(a\sec\theta, b\tan\theta)$

$x = a\sec\theta$ $\qquad$ $y = b\tan\theta$

D w.r.to θ $\qquad$ D w.r.to θ

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ $\qquad$ $\frac{dy}{d\theta} = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\frac{\sec\theta}{\tan\theta}$$

$$\frac{dy}{dx} = \frac{b}{a} \cdot \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta}$$

$$\frac{dy}{dx} = \frac{b}{a}\operatorname{cosec}\theta$$

D w.r.to x

$$\frac{d^2y}{dx^2} = \frac{b}{a}(-\operatorname{cosec}\theta\cot\theta) \cdot \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\operatorname{cosec}\theta\cot\theta\,\frac{1}{a\sec\theta\tan\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\,\cancel{\mathrm{cosec}}\,\frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta} \cdot \frac{\cos\theta \cdot \cos\theta}{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b\cos^3\theta}{a^2\sin^3\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$$

$y_1 = +\frac{b}{a}\operatorname{cosec}\theta$; $y_2 = \frac{-b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$

Let (X, Y) be the centre of curvature where

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$X = a\sec\theta - \frac{\frac{b}{a}\operatorname{cosec}\theta\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{\frac{-b\cos^3\theta}{a^2\sin^3\theta}}$$

$$X = a\sec\theta + \frac{\cancel{b}}{\cancel{a}}\frac{1}{\sin\theta}\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)\frac{a^2\sin^3\theta}{\cancel{b}\cos^3\theta}$$

$$X = a\sec\theta + \frac{a\sin^2\theta}{\cos^3\theta} + \frac{b^2a}{a^2}\cdot\frac{1}{\sin^2\theta}\cdot\frac{\sin^2\theta}{\cos^3\theta}$$

$$X = a\sec\theta + \frac{a\sin^2\theta}{\cos^2\theta}\frac{1}{\cos\theta} + \frac{b^2}{a}\sec^3\theta$$

$$X = a\sec\theta + a\tan^2\theta(\sec\theta) + \frac{b^2}{a}\sec^3\theta$$

~~X = a sec³θ +~~

$$X = a\sec\theta(1+\tan^2\theta) + \frac{b^2}{a}\sec^3\theta$$

$$X = a\sec^3\theta + \frac{b^2}{a}\sec^3\theta$$

$$X = \sec^3\theta\left(a + \frac{b^2}{a}\right)$$

$$X = \sec^3\theta\left(\frac{a^2+b^2}{a}\right) \quad -①$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= b\tan\theta + \frac{\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{\frac{-b\cos^3\theta}{a^2\sin^3\theta}}$$

$$Y = b\tan\theta - \frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta - \frac{b\sin^3\theta}{\cos^3\theta}\cdot\frac{1}{\sin^2\theta}$$

$$= b\tan\theta - \frac{a^2}{b^2}\tan^3\theta - \frac{b\sin\theta}{\cos\theta}\cdot\frac{1}{\cos^2\theta}$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta - b\tan\theta\sec^2\theta$$

$$= b\tan\theta - b\tan\theta\sec^2\theta - \frac{a^2}{b}\tan^3\theta$$

$$= b\tan\theta(1-\sec^2\theta) - \frac{a^2}{b}(\tan^3\theta)$$

$$= b\tan\theta(-\tan^2\theta) - \frac{a^2}{b}(\tan^3\theta)$$

$= -b\tan^3\theta - \frac{a^2}{b}(\tan^3\theta)$

$= -\tan^3\theta\left(\frac{b^2+a^2}{b}\right)$

$x = \sec^3\theta\left(\frac{a^2+b^2}{a}\right)$ $\qquad$ $y = -\tan^3\theta\left(\frac{b^2+a^2}{b}\right)$

$ax = \sec^3\theta(a^2+b^2)$ $\qquad$ $yb = -(a^2+b^2)\tan^3\theta$

$(ax)^{2/3} = (a^2+b^2)^{2/3}(\sec^3\theta)^{2/3}$ $\qquad$ $(by)^{2/3} = (a^2+b^2)^{2/3}(\tan^3\theta)^{2/3}$

$(by)^{2/3} = (a^2+b^2)^{2/3}\tan^2\theta$

$(ax)^{2/3} = (a^2+b^2)^{2/3}\sec^2\theta$

$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}(\sec^2\theta - \tan^2\theta)$

$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$

$\therefore$ The evolute is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$ //

Envelope:

Working Rule:-

Given $f(x, y, z)$ where α is the parameter

Differentiate partially wrt α

Eliminating α from the given eqn then that eqn is called envelope equation.

Ex: find the envelope $x\cos\alpha + y\sin\alpha = l\sin\alpha\cos\alpha$ where α is the parameter.

Given eqn $x\cos\alpha + y\sin\alpha = l\sin\alpha\cos\alpha$ — (1)

Divide b.s by $\sin\alpha\cos\alpha$

$$\frac{x\cos\alpha}{\sin\alpha\cos\alpha} + \frac{y\sin\alpha}{\sin\alpha\cos\alpha} = \frac{l\sin\alpha\cos\alpha}{\sin\alpha\cos\alpha}$$

$$\frac{x}{\sin\alpha} + \frac{y}{\cos\alpha} = l \text{ — (1)}$$

P.D w.r.to α

$$x\left(\frac{1}{\sin^2\alpha}\right)(\cos\alpha) + y\left(\frac{-1}{\cos^2\alpha}\right)(-\sin\alpha) = 0$$

$$\frac{y\sin\alpha}{\cos^2\alpha} = \frac{x\cos\alpha}{\sin^2\alpha}$$

$$y\sin^3\alpha = x\cos^3\alpha$$

$$\frac{\sin^3\alpha}{\cos^3\alpha} = \frac{x}{y}$$

$$\frac{\sin\alpha}{\cos\alpha} = \left(\frac{x}{y}\right)^{1/3}$$

$$\tan\alpha = \left(\frac{x}{y}\right)^{1/3}$$

ind envelope of

$x^2\sin\alpha + y^2\cos\alpha = a^2$ where α is parameter.

Given,

$x^2\sin\alpha + y^2\cos\alpha = a^2$ — (1)

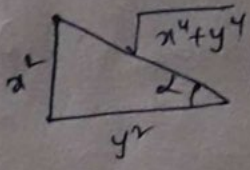
P.D w.r.to x

$x^2\cos\alpha + y^2(-\sin\alpha) = 0$

$x^2\cos\alpha - y^2\sin\alpha = 0$

$x^2\cos\alpha = y^2\sin\alpha$

$\dfrac{x^2}{y^2} = \dfrac{\sin\alpha}{\cos\alpha}$



$\tan\alpha = \dfrac{x^2}{y^2}$

$\sin\alpha = \dfrac{x^2}{\sqrt{x^4+y^4}}$ $\quad\cos\alpha = \dfrac{y^2}{\sqrt{x^4+y^4}}$

$\sin\alpha$ and $\cos\alpha$ value sub in eq (1)

$x^2\left(\dfrac{x^2}{\sqrt{x^4+y^4}}\right) + y^2\left(\dfrac{y^2}{\sqrt{x^4+y^4}}\right) = a^2$

$\dfrac{x^4+y^4}{\sqrt{x^4+y^4}} = a^2$

$\dfrac{\sqrt{x^4+y^4}\ \sqrt{x^4+y^4}}{\sqrt{x^4+y^4}} = a^2$

$$\sqrt{x^4+y^4} = a^2$$

S.onB

$$\boxed{x^4+y^4 = a^4}$$

Ex: find the envelope of $y^2 = m^2(x-m)$ where m is the parameter

Given eq^n $y^2 = m^2x - m^3$ — (1)

P.D w.r.to m

$$2mx - 3m^2 = 0$$

$$2\cancel{m}x = 3\cancel{m}^2$$

$$2x = 3m$$

$$m = \frac{2x}{3}$$

m value sub in eq (1)

$$y^2 = \frac{4x^2}{9}(x) - \frac{8x^3}{27}$$

$$y^2 = \frac{4x^3}{9}\left(1 - \frac{2}{3}\right)$$

$$y^2 = \frac{4x^3}{9}\left(\frac{1}{3}\right)$$

$$y^2 = \frac{4x^3}{27}$$

$$\therefore 27y^2 = 4x^3$$

7) Find the envelope of $y = mx + \frac{a}{m}$, where m is parameter.

Given $y = mx + \frac{a}{m}$ — (1)

P.D w.r.to m

$$0 = x + a\left(\frac{-1}{m^2}\right)$$

$$x = \frac{a}{m^2}$$

$$m^2 = \frac{a}{x}$$

$$m = \sqrt{\frac{a}{x}}$$

sub m in eq (1)

$$y = \sqrt{\frac{a}{x}}\, x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$y = \sqrt{\frac{a}{x}}\sqrt{x}\sqrt{x} + \sqrt{a}\cdot\sqrt{\frac{x}{a}}$$

$$y = \sqrt{ax} + \sqrt{ax} = 2\sqrt{ax}$$

$$\boxed{y^2 = 4ax.}$$

Find the envelope eqn $y = mx + am^3$ where m is parameter

Given eqn $y = mx + am^3$ — (1)

P.D w.r.to m

$$0 = x + 3am^2$$

$$3am^2 = x$$

$$m^2 = \frac{-x}{3a}$$

$$y = mx + am^3$$

S.O.B.S

$$y^2 = (mx + am^3)^2$$

$$y^2 = m^2x^2 + a^2m^6 + 2m^4ax$$

$$y^2 = \left(\frac{-x}{3a}\right)(x^2) + \cancel{a^2}\left(\frac{-x}{3\cancel{a}}\right)\left(\frac{-x}{3\cancel{a}}\right)\left(\frac{-x}{3a}\right) + 2\left(\frac{-x}{3a}\right)^2 ax$$

$$= \frac{-x^3}{3a} - \frac{x^3}{27a} + \frac{2x^3}{9a}$$

$$y^2 = \frac{-9x^3 - x^3 + 6x^3}{27a}$$

$$27ay^2 = -4x^3$$

$$\boxed{4x^3 + 27ay^2 = 0}$$

Method no-2

If A, B, C are the functions of x and y, m is a parameter then the envelope of $Am^2+Bm+C=0$ is $B^2=4AC$

Given eqn is $Am^2+Bm+C=0$ — (1)

P.D w.r.to m

$$2m \cdot A+B=0$$

$$2Am=-B$$

$$M=\frac{-B}{2A}$$

sub M in eq (1)

$$A\left(\frac{-B}{2A}\right)^2+B\left(\frac{-B}{2A}\right)+C=0$$

$$\frac{\cancel{A}B^2}{4A^{\cancel{2}}}-\frac{B^2}{2A}+C=0$$

$$\frac{B^2-2B^2+4AC}{4A}=0$$

$$-B^2+4AC=0$$

$$B^2=AC4$$

$$\boxed{B^2=4AC}$$ //

1) Find the $y=mx+\frac{a}{m}$ where m is the parameter

Given eqn $y=mx+\frac{a}{m}$

$$my=m^2x+a$$

$m^2 x - ym + a = 0$

$x(m^2) + (-y)m + a = 0$

Above eqn compare with $Am^2 + Bm + C$.

$A = x,\ B = -y,\ C = a$

eqn of envelope

$B^2 = 4AC$

$\boxed{y^2 = 4ax}$

(3) find the envelope of $y = mx + \frac{1}{m}$

Given eqn is $y = mx + \frac{1}{m}$

$y = \frac{m^2 x + 1}{m}$

$ym = m^2 x + 1$

$m^2 x - ym + 1 = 0$

$xm^2 - ym + 1 = 0$

Compare $Am^2 + Bm + C = 0$

$A = x \mid B = -y \mid C = 1$

eqn of envelope

$B^2 = 4AC$

$y^2 = 4(x)(1)$

$\boxed{y^2 = 4x}$

) find the envelope of the curve $\frac{x^2}{a^2}+\frac{y^2}{k^2-\alpha^2}=1$ where α is parameter

G. eqⁿ $\frac{x^2}{\alpha^2}+\frac{y^2}{k^2-\alpha^2}=1$.

$$\frac{x^2(k^2-\alpha^2)+y^2\alpha^2}{\alpha^2(k^2-\alpha^2)}=1$$

$$x^2(k^2-\alpha^2)+y^2\alpha^2=\alpha^2k^2-\alpha^4$$

$$x^2k^2-x^2\alpha^2+y^2\alpha^2-\alpha^2k^2+\alpha^4=0$$

$$\alpha^4+(y^2-x^2-k^2)\alpha^2+x^2k^2=0$$

$$(\alpha^2)^2+(y^2-x^2-k^2)\,\alpha^2+x^2k^2=0$$

$A=1 \quad B=y^2-x^2-k^2. \quad C=x^2k^2$

eqⁿ of envelope

$$B^2=4AC$$

$$(y^2-x^2-k^2)^2=4(1)(x^2k^2)$$

) find the envelope $y=mx+\sqrt{a^2m^2+b^2}$ where m is parameter

Given $y=mx+\sqrt{a^2m^2+b^2}$

S.O.B.S

$$(y-mx)=\sqrt{a^2m^2+b^2}$$

$$(y-mx)^2=a^2m^2+b^2$$

$y^2+m^2x^2-2mxy=a^2m^2+b^2$

$y^2+m^2x^2-2xmy-a^2m^2-b^2=0$

$(x^2-a^2)m^2-(2xy)m+y^2-b^2=0$

$A=x^2-a^2,\ B=-2xy,\ C=y^2-b^2$

eq^n of envelope $B^2=4AC$

$4x^2y^2=4(x^2-a^2)(y^2-b^2)$

$4x^2y^2=4x^2-4a^2(y^2-b^2)$

$4x^2y^2=4x^2y^2-4x^2b^2-4a^2y^2+4a^2b^2$

$a^2b^2-a^2y^2-x^2b^2=0$

(6) find the eq^n of envelope $y=mx+a\sqrt{1+m^2}$

Given $y=mx+a\sqrt{1+m^2}$

$y-mx=a\sqrt{1+m^2}$

S.OBS

$(y-mx)^2=a^2(1+m^2)$

$y^2+m^2x^2-2xym=a^2+a^2m^2$

$m^2x^2-a^2m^2+y^2-2xym-a^2=0$

$m^2(x^2-a^2)-2xym+(y^2-a^2)=0$

compare $Am^2+Bm+C=0$

$A=x^2-a^2 \qquad B=-2xy \quad C=y^2-a^2$

$B^2 = 4AC$

$4x^2y^2 = 4(x^2-a^2)(y^2-a^2)$

$\boxed{y^2 = a^2 - x^2}$

$x^2y^2 = (x^2-a^2)(y^2-a^2)$

$x^2y^2 = x^2y^2 - x^2a^2 - a^2y^2 + a^4$

$x^2a^2 + a^2y^2 - a^4 = 0$

$a^2(x^2+y^2-a^2) = 0$

find the eqⁿ of evolute $(x-\alpha)^2 + y^2 = 4\alpha$

Given eqⁿ $(x-\alpha)^2 + y^2 = 4\alpha$

$x^2 + \alpha^2 - 2x\alpha + y^2 = 4\alpha$

$\alpha^2 - 2x\alpha - 4\alpha + x^2 + y^2 = 0$

$\alpha^2 - 2\alpha(x+2) + (x^2+y^2) = 0$

compare, $Am^2 + Bm + C = 0$

$A = 1 \mid B = -2(x+2) \mid C = x^2+y^2$

$B^2 = 4AC$

$4(x+2)^2 = 4(1)(x^2+y^2)$

$4(x^2+4+4x) = 4(x^2+y^2)$

$x^2 - x^2 + 4x + 4 - y^2 = 0$

$y^2 - 4x - 4 = 0$

$y^2 = 4(x+1)$!!

(8) find the eq^n of evolute of $y^2 = m^2x + \frac{1}{m^2}$ where m is a parameter.

Given $y^2 = m^2x + \frac{1}{m^2}$

$$m^2y = m^4x + 1$$

$$m^4x - m^2y + 1 = 0$$

$$(m^2)^2x - ym^2 + 1 = 0$$

$$(x)(m^2)^2 - ym^2 + 1 = 0$$

compare $Am^2 + Bm + C = 0$

$$A = x \mid B = -y \mid C = 1$$

$$B^2 = 4AC$$

$$y^2 = 4x \,//$$

(9) find the equation of evolute :- $(x-\alpha)^2 + (y-\alpha)^2 = 2\alpha$ where α is an parameter.

Given eq^n $(x-\alpha)^2 + (y-\alpha)^2 = 2\alpha$

$$x^2 + \alpha^2 - 2x\alpha + y^2 + \alpha^2 - 2y\alpha - 2\alpha = 0$$

$$2\alpha^2 - 2(x+y+1)\alpha + y^2 + x^2 = 0$$

Compare $A\alpha^2 + B\alpha + 1 = 0$

$$A = 2 \mid B = -2(x+y+1) \mid C = x^2 + y^2$$

$$B^2 = 4AC$$

$$\cancel{4}(x+y+1)^2 = \cancel{4}(x^2+y^2)2$$

$$x^2+y^2+1+2x+2y+2xy = 2(x^2+y^2)$$

$$2x^2+2y^2-x^2-y^2-2x-2y-2xy-1=0$$

$$x^2+y^2-2x-2y-2xy-1=0$$